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WAVE REFLECTION

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ABSTRACT

This report is Aalborg University's first contribution to the MAS2–CT92 project: *Full scale dynamic load monitoring of rubble mound breakwaters.*

One of the objectives of the MAST II programme is determination of the forces acting on the armour units and the hydraulic pressure set-up in the breakwater core and foundation layers.

In order to interpret the measured forces correct, it is important to know the incident waves to the breakwater. As the wave field in front of a reflecting structure consist of both incident waves and reflected waves, methods capable of separating the incident waves and the reflected waves must be used.

This report introduces the wave reflection phenomenon, presents methods for estimating wave reflection and gives status for ongoing evaluation of the methods for estimating reflection.

Finally, a appropriate method to be used in the MAST II programme in order to describe incident waves is proposed.

1 Introduction

1.1 Design of harbours and breakwaters

Coastal and offshore engineering cover a wide spectrum in the field of civil engineering. Some of the major subjects involved are the construction of harbours and breakwaters.

Design of breakwaters and breakwater lay-outs are based on the need for an acceptable wave climate in the harbour and in the harbour entrance. However, the complex behaviour of waves propagating into a more or less closed basin makes the design of the quaywalls and breakwaters important.

When a physical or numerical model is used for wave climate investigations it is important that the model will provide reliable results. Naturally this depends upon whether the boundary conditions in the model are correct.

Establishing the correct boundary conditions is a problem due to scale effects. However the important properties of a boundary are the absorption, transmission and reflection of waves.

In order to provide correct reflection characteristics in the models one must know the true reflection characteristics for the structure under study.

The wave field in front of a structure consist of both incident waves and reflected waves. As one cannot first measure the incident waves and then the reflected waves, methods are required to separate or distinguish between the incident and reflected waves.

1.2 Wave Reflection

Wave reflection occurs when waves are propagating onto breakwaters and quaywalls.

Separating the incident waves and the reflected waves from a recorded wave elevation time series has several purposes. Either:

- The reflection characteristic for a structure is wanted, or
- The incident wave characteristic for a given site is wanted

The reflection characteristics for a structure are mainly wanted in sheltered areas, e.g. inside harbours, where the level of wave disturbance plays an important role in the design situation. Though, also the reflection characteristics of breakwaters are wanted, because it is important for the wave climate in the harbour entrance.

Generally a low reflection is wanted.

In order to apply physical models and numerical models the reflection characteristics must be modelled correct.

The incident waves are wanted where response of a structure (breakwater) is wanted as function of the incoming waves in front (seaward) of the reflecting structure.

In the following several different methods for separating incident waves and reflected waves will be presented. After separation of the incident waves and the reflected waves it is easy to calculate the reflection characteristics for the structure.

The following methods for separating the waves will be divided into two main groups:

- 2D methods
  The waves are assumed to be 2-dimensional (long crested)
- 3D methods
  The waves are assumed to be 3-dimensional (short crested)

There is a wide need for 2D methods due to the amount of experiments carried out in wave flumes. In wave flumes reflection and re-reflection influence upon the validity of measurements if it is not possible to extract the incident waves. Further when operating in the near shore environment the wave-field will often be approximately two-dimensional as the waves diffract, i.e. they bend towards orthogonals to the shore.

The 2D methods presented herein are all based on surface elevation measurements at a number of positions.

Three of the 2D methods are derived from the same principles but for various number of probes. The simplest method requires two probes. The other methods need three or more probes. The three first methods work in *frequency domain* and give the incident wave spectrum as well as the reflected wave spectrum. The last 2D method works in *time domain* and gives the incident wave timeseries.

There is a considerable step from 2D methods to 3D methods. This is due to the circumstance that 3D methods for estimation of random sea states even without reflection are tedious and not as reliable as 2D methods. In principle, all methods for estimating directional wavespectra may be considered as a tool to determine reflection coefficients. However, only a few methods are capable of handling reflected seas as the reflected waves propagate with the same frequency as the incident waves, and thereby makes it impossible or at least difficult to distinguish. One should also recall that the direction of the waves is unknown in contrast to a 2D case.

As an intermediate case there is oblique long crested waves. This may be considered as a 3D case having a very narrow peak in the directional spreading function. This justifies the assumption that only one direction is present. Furthermore in laboratories real oblique waves can be generated. This may be used to determine the 3D reflection characteristics for e.g. a breakwater by use of physical models.

2 Two-dimensional methods

In the following four methods for separation of incident and reflected waves in a two-dimensional wave field will be presented.

The first three methods are all working in the *frequency domain* and have the same basic principle. They assume the wave elevation to be a sum of regular waves travelling with different frequency and phase. The first method by Goda & Suzuki needs measurements of the wave elevation in two distinct points. Hence by use of Fourier analysis the amplitude of the incident and reflected waves for a given frequency can be estimated. This procedure does not account for the noise which probably is contaminated in the measured wave signals, i.e. the measured wave elevations.

The method presented by Mansard & Funke takes this into account, but also requires the wave elevation to be measured in three distinct points. By applying Fourier analysis this should result in the same waves except that the measured noise varies from wave gauge to wave gauge. That is, 10 Fourier coefficients are needed but only 6 are available. Instead the noise is expressed in terms of the Fourier coefficients for the incident and reflected waves and the measured coefficients. The best estimate of the Fourier coefficients is then found by minimising the noise.

The method by Zelt & Skjelbreia extents the method to apply to an arbitrary number (though larger than one) of wave gauges. Further a weighting of each frequency component is introduced. This is to control the possible influence of singularities which occur for some geometric relations between the distances between the probes.

Finally a method by Frigaard & Brorsen is presented. This method is based on another approach than the previously mentioned methods. The method has the advantage that it works in the *time domain*.

Other methods have been published but they will not be presented herein. The methods by Goda & Suzuki and Mansard & Funke are the methods being used most often as they are relatively simple to apply and yield reliable results within most applications.

2.1 Goda & Suzuki's method

The method presented by Goda & Suzuki (1976) is the most simple method. It is based on the assumption that the wave elevation can be considered as a sum of waves travelling with different frequency, amplitude and phase. Further for each wave a reflected wave will travel in the opposite direction. The method makes use of Fourier analysis and will due to singularities put constraint to the distance between the waveprobes. The method is very easy to implement but has a lack of accuracy as no measuring errors are accounted for.

The surface elevation in a two-dimensional wave field is assumed to be a summation of a number of waves, say N waves, i.e.

$$\eta(x,t) = \sum_{n=1}^{N} a_n \cos(k_n x - \omega_n t + \Phi_n) \tag{2.1}$$

where η is the surface elevation relative to MWL
x is the position of the wave gauge in a predefined coordinatesystem
t is time
a_n is the amplitude
k_n is the wavenumber
ω_n is the angular frequency of the waves
Φ_n is the phase

If reflection happens ?? may be expanded to

$$\eta(x,t) = \sum_{n=1}^{N} a_{I,n} \cos(k_n x - \omega_n t + \Phi_{I,n}) + \sum_{n=1}^{N} a_{R,n} \cos(k_n x + \omega_n t + \Phi_{R,n}) \tag{2.2}$$

where indices I and R denotes incident and reflected respectively.

In the following the index n will be omitted for simplicity, that is, only one frequency is considered. Hence eq. (??) will consist of only two terms:

$$\eta(x,t) \quad = \quad a_I \cos(kx - \omega t + \Phi_I) + a_R \cos(kx + \omega t + \Phi_R) \tag{2.3}$$

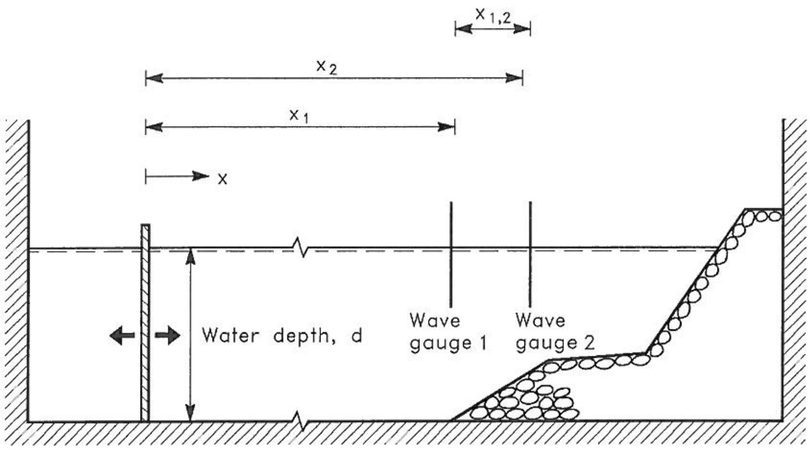

Figure 2.1: Definition sketch showing $x_{1,2}$.

Sampling the surface elevation at two positions within a distance of $x_{1,2}$ measured in the direction of propagation of the waves, as illustrated in fig. ??, yields:

$$\begin{aligned} \eta_1 = \eta(x_1, t) &= a_I \cos(kx_1 - \omega t + \Phi_I) + a_R \cos(kx_1 + \omega t + \Phi_R) \\ \eta_2 = \eta(x_2, t) &= a_I \cos(kx_2 - \omega t + \Phi_I) + a_R \cos(kx_2 + \omega t + \Phi_R) \end{aligned} \tag{2.4}$$

Decomposing the trigonometric terms in eq. (??) by use of

$$\cos A \cos B \pm \sin A \sin B = \cos(A \mp B) \tag{2.5}$$

leads to

$$\begin{aligned} \eta_1 &= a_I \left(\sin(kx_1 + \Phi_I)\sin(\omega t) + \cos(kx_1 + \Phi_I)\cos(\omega t)\right) \\ &\quad + a_R \left(\cos(kx_1 + \Phi_R)\cos(\omega t) - \sin(kx_1 + \Phi_R)\sin(\omega t)\right) \end{aligned} \tag{2.6}$$

$$\begin{aligned} \eta_2 &= a_I \left(\sin(kx_2 + \Phi_I)\sin(\omega t) + \cos(kx_2 + \Phi_I)\cos(\omega t)\right) \\ &\quad + a_R \left(\cos(kx_2 + \Phi_R)\cos(\omega t) - \sin(kx_2 + \Phi_R)\sin(\omega t)\right) \end{aligned} \tag{2.7}$$

which is rearranged to

$$\begin{aligned} \eta_1 &= A_1 \cos(\omega t) + B_1 \sin(\omega t) \\ \eta_2 &= A_2 \cos(\omega t) + B_2 \sin(\omega t) \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} A_1 &= a_I \cos(kx_1 + \Phi_I) + a_R \cos(kx_1 + \Phi_R) \\ B_1 &= a_I \sin(kx_1 + \Phi_I) - a_R \sin(kx_1 + \Phi_R) \\ A_2 &= a_I \cos(kx_2 + \Phi_I) + a_R \cos(kx_2 + \Phi_R) \\ B_2 &= a_I \sin(kx_2 + \Phi_I) - a_R \sin(kx_2 + \Phi_R) \end{aligned} \tag{2.9}$$

In eq. (??) the elevation is seen to be a composite signal of a sine and cosine signal having different time-constant amplitude, i.e. the LHS of eq. (??) must correspond to the Fourier coefficients which can be obtained from Fourier analysis of the recorded time series.

Thus eq. (??) contains four equations with four unknowns, i.e. a_I, a_R, Φ_I and Φ_R. The solution giving the amplitudes is given by Goda & Suzuki (1976) as:

$$
\begin{aligned}
a_I &= \frac{1}{2|\sin(kx_{1,2})|}\sqrt{\begin{array}{l}(A_2 - A_1\cos(kx_{1,2}) - B_1\sin(kx_{1,2}))^2 \\ +(B_2 + A_1\sin(kx_{1,2}) - B_1\cos(kx_{1,2}))^2\end{array}} \\
a_R &= \frac{1}{2|\sin(kx_{1,2})|}\sqrt{\begin{array}{l}(A_2 - A_1\cos(kx_{1,2}) + B_1\sin(kx_{1,2}))^2 \\ +(B_2 - A_1\sin(kx_{1,2}) - B_1\cos(kx_{1,2}))^2\end{array}}
\end{aligned}
\tag{2.10}
$$

where $x_{1,2} = x_2 - x_1$.

It is seen from eq. (??) that singularities will occur for $\sin(kx_{1,2}) = 0$. Hence it should be avoided that

$$\frac{x_{1,2}}{L} = \frac{n}{2} \qquad \text{where} \qquad n = 0, 1, 2, ...$$

Further Goda & Suzuki (1976) suggest to avoid values in the range $\pm 0.05\frac{x_{1,2}}{L}$ at the singularity points. For a wide wave spectrum this will be impossible for all frequencies, but of course values applying for the peak frequencies should be weighted highest.

For regular waves the method is quite accurate but for irregular waves the confidence of the FFT analysis plays a significant role. However in both cases noise may be the dominant error as it cannot be accounted for.

2.2 Mansard & Funke's method

As a natural extension to the method by Goda & Suzuki, Mansard & Funke (1980) presented a three-points method. Here an additional probe is taken into use which makes it possible to add an error to the measurements and hence minimise it in a least squares sense.

The general equation for a progressive wave field is obtained as in the previous method, i.e. eq. (??):

$$\eta(x,t) = \sum_{n=1}^{N} a_n \cos(k_n x - \omega_n t + \Phi_n) \tag{2.11}$$

The wave elevation given by eq. (??) and eq. (??) is separated into incident waves and reflected waves, also as previously, but now a noise function is added. This leads to:

$$\eta(x,t) = \sum_{n=1}^{N} a_{I,n} \cos(k_n x - \omega_n t + \Phi_{I,n}) + \sum_{n=1}^{N} a_{R,n} \cos(k_n x + \omega_n t + \Phi_{R,n})$$

or

$$\begin{aligned} \eta(x,t) \quad &= \quad \sum_{n=1}^{N} a_{I,n} \cos(k_n x - \omega_n t + \Phi_n) \\ &\quad + \sum_{n=1}^{N} a_{R,n} \cos(k_n(x + 2x_R) + \omega_n t + \Phi_n + \theta_s) + \Omega(t) \end{aligned} \tag{2.12}$$

Here in contrary to Goda & Suzuki (1976) the distance from the point of observation to the reflecting structure is assumed known. However as the distance may be impossible to determine (e.g. for a slope) a phase θ_s is introduced to compensate for this. This has the consequence that the phase Φ_n remains the same for the incident and reflected wave. $\Omega(t)$ is the noise function and expresses all kinds of errors. x_R is the distance from the wave probe to the reflecting structure.

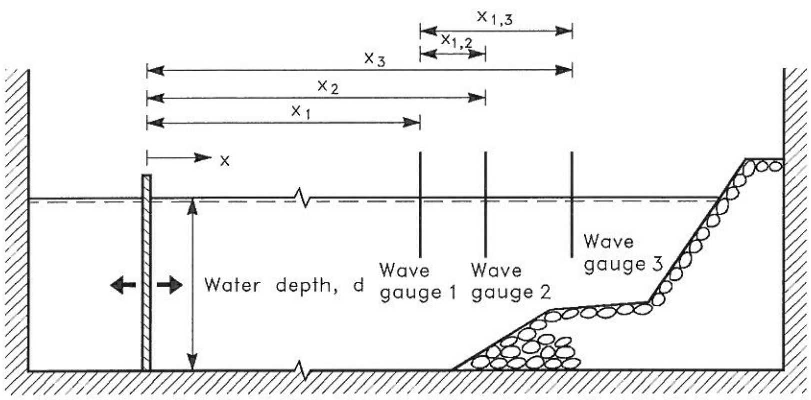

Figure 2.2: Definition sketch.

Mansard & Funke allow a phaseshift θ_s to occur at the reflecting structure. For three probes placed as indicated on fig. ?? eq. (??) can be applied for each probe.

Hence

$$\begin{aligned}
\eta_p &= \eta(x_p, t) \\
&= \sum_{n=1}^{N} a_{I,n} \cos(k_n x_p - \omega_n t + \Phi_n) \\
&\quad + \sum_{n=1}^{N} a_{R,n} \cos(k_n(x_p + 2x_{R,p}) + \omega_n t + \Phi_n + \theta_s) + \Omega_p(t) \qquad (2.13)
\end{aligned}$$

where index p refer to probe number.

Inserting $x_{1,p}$, which is the distance from the 1st probe to the p'th probe, eq. (??) can be modified to

$$\begin{aligned}
\eta_p &= \sum_{n=1}^{N} a_{I,n} \cos(k_n(x_1 + x_{1,p}) - \omega_n t + \Phi_n) \\
&\quad + \sum_{n=1}^{N} a_{R,n} \cos(k_n(x_1 + 2x_{R,1} - x_{1,p}) + \omega_n t + \Phi_n + \theta_s) + \Omega_p(t) \qquad (2.14)
\end{aligned}$$

In the following until eq. (??) only one frequency will be considered, i.e. index n will be omitted. Thus

$$\begin{aligned}
\eta_p &= a_I \cos(k(x_1 + x_{1,p}) - \omega t + \Phi) \\
&\quad + a_R \cos(k(x_1 + 2x_{R,1} - x_{1,p}) + \omega t + \Phi + \theta_s) + \Omega_p(t) \qquad (2.15)
\end{aligned}$$

The following manipulations have the purpose to derive an algebraic solution, which is based on a minimisation of the noise function.

Fourier transformation of eq. (??) yields

$$\begin{aligned}
A_p + iB_p &= a_I \exp\left(ik(x_1 + x_{1,p}) + i\Phi\right) \\
&\quad + a_R \exp\left(ik(x_1 + 2x_{R,1} - x_{1,p}) + i(\Phi + \theta_s)\right) \\
&\quad + Y_p \exp(i\rho_p) \qquad (2.16)
\end{aligned}$$

where A_p and B_p are the Fourier coefficients.
Let

$$Z_I = a_I \exp(ikx_1 + i\Phi) \qquad (2.17)$$

$$Z_R = a_R \exp(ik(x_1 + 2x_{R,1}) + i(\Phi + \theta_s)) \qquad (2.18)$$

$$Z_{N,p} = Y_p \exp(i\rho_p) \qquad (2.19)$$

where index N refer to noise.

Now, eq. (??) can be applied to each probe yielding

$$\begin{aligned}
A_1 + iB_1 &= Z_I + Z_R + Z_{N,1} \\
A_2 + iB_2 &= Z_I \exp(ikx_{1,2}) + Z_R \exp(-ikx_{1,2}) + Z_{N,2} \\
A_3 + iB_3 &= Z_I \exp(ikx_{1,3}) + Z_R \exp(-ikx_{1,3}) + Z_{N,3}
\end{aligned}$$

or in general form

$$A_p + iB_p = Z_I \exp\left(ikx_{1,p}\right) + Z_R \exp\left(-ikx_{1,p}\right) + Z_{N,p} \tag{2.20}$$

One is interested in solving ?? with regard to Z_I and Z_R as they contain information of the reflection coefficients.

Now eq. (??) can be rearranged to

$$\begin{aligned}
\varepsilon_1 &= Z_I + Z_R - (A_1 + iB_1) \\
\varepsilon_2 &= Z_I \exp(ikx_{1,2}) + Z_R \exp(-ikx_{1,2}) - (A_2 + iB_2) \\
\varepsilon_3 &= Z_I \exp(ikx_{1,3}) + Z_R \exp(-ikx_{1,3}) - (A_3 + iB_3)
\end{aligned} \tag{2.21}$$

where

$$\varepsilon_p = -Z_{N,p} + f_e(Z_I, Z_R)$$

Minimising the noise function $\Omega_p(t)$ introduced in eq. (??) correspond to minimising the sum of squares of ε_p for all p, i.e.

$$\sum_{p=1}^{3} (\varepsilon_p)^2 = \sum_{p=1}^{3} (Z_I \exp(ikx_{1,p}) + Z_R \exp(-ikx_{1,p}) - (A_p + iB_p))^2 \tag{2.22}$$

should be minimised.

Assuming that the minimum of eq. (??) is achieved when both partial derivatives are zero, i.e.

$$\frac{\partial \sum_{p=1}^{3} \varepsilon_p^2}{\partial Z_I} = \frac{\partial \sum_{p=1}^{3} \varepsilon_p^2}{\partial Z_R} = 0 \tag{2.23}$$

Hence one obtains:

$$\begin{aligned}
0 &= 2 \sum_{p=1}^{3} (Z_I \exp(ikx_{1,p}) + Z_R \exp(-ikx_{1,p}) - (A_p + iB_p)) \exp(ikx_{1,p}) \\
0 &= 2 \sum_{p=1}^{3} (Z_I \exp(ikx_{1,p}) + Z_R \exp(-ikx_{1,p}) - (A_p + iB_p)) \exp(-ikx_{1,p})
\end{aligned} \tag{2.24}$$

When written out and again including index n eq. (??) leads to

$$\begin{aligned}
Z_{I,n}(1+\exp(i2k_nx_{1,2})+\exp(i2k_nx_{1,3}))+3Z_{R,n} &= \\
(A_{1,n}+iB_{1,n})+(A_{2,n}+iB_{2,n})\exp(ik_nx_{1,2})+(A_{3,n}+iB_{3,n})\exp(ik_nx_{1,3}) \\
Z_{R,n}(1+\exp(-i2k_nx_{1,2})+\exp(-i2k_nx_{1,3}))+3Z_{I,n} &= \\
(A_{1,n}+iB_{1,n})+(A_{2,n}+iB_{2,n})\exp(-ik_nx_{1,2})+(A_{3,n}+iB_{3,n})\exp(-ik_nx_{1,3})
\end{aligned} \tag{2.25}$$

The solution is given by Mansard & Funke (1980) as:

$$\begin{aligned}
Z_{I,n} &= \tfrac{1}{D_n}((A_{1,n}+iB_{1,n})(R_{1,n}+iQ_{1,n}) \\
&\quad +(A_{2,n}+B_{2,n})(R_{2,n}+iQ_{2,n})+(A_{3,n}+iB_{3,n})(R_{3,n}+iQ_{3,n})) \\
Z_{R,n} &= \tfrac{1}{D_n}((A_{1,n}+iB_{1,n})(R_{1,n}-iQ_{1,n}) \\
&\quad +(A_{2,n}+iB_{2,n})(R_{2,n}-iQ_{2,n})+(A_{3,n}+iB_{3,n})(R_{3,n}-iQ_{3,n}))
\end{aligned} \tag{2.26}$$

where

$$\begin{aligned}
D_n &= 2(\sin^2(k_nx_{1,2})+\sin^2(k_nx_{1,3})+\sin^2((k_nx_{1,3})-k_nx_{1,2})) \\
R_{1,n} &= \sin^2(k_nx_{1,2})+\sin^2(k_nx_{1,3}) \\
Q_{1,n} &= \sin(k_nx_{1,2})\cos(k_nx_{1,2})+\sin(k_nx_{1,3})\cos(k_nx_{1,3}) \\
R_{2,n} &= \sin(k_nx_{1,3})\sin(k_nx_{1,3}-k_nx_{1,2}) \\
Q_{2,n} &= \sin(k_nx_{1,3})\cos(k_nx_{1,3}-k_nx_{1,2})-2\sin(k_nx_{1,2}) \\
R_{3,n} &= -\sin(k_nx_{1,2})\sin(k_nx_{1,3}-k_nx_{1,2}) \\
Q_{3,n} &= \sin(k_nx_{1,2})\cos(k_nx_{1,3}-k_nx_{1,2})-2\sin(k_nx_{1,3})
\end{aligned}$$

The only unknowns in eq. (??) are the Fourier coefficients of the measured wave elevations. These are obtained by use of FFT analysis of the measurements.

Compared to the previously discussed method this method has the advantage that it minimises the noise contaminated to the elevation measurements. Thus instead of obtaining an exact solution a fitted solution is obtained.

Problems due to singularities, however, occur as well. Singularity will occur when

$$D_n = 0$$

i.e.

$$\sin^2(k_nx_{1,2})+\sin^2(k_nx_{1,3})+\sin^2(k_nx_{1,3}-k_nx_{1,2}) = 0$$

As all the terms are positive the solution is solution to

$$\sin(k_n x_{1,2}) = \sin(k_n x_{1,3}) = \sin(k_n x_{1,3} - k_n x_{1,2}) = 0$$

but if $\sin(k_n x_{1,2}) = \sin(k_n x_{1,3}) = 0$ then $\sin(k_n x_{1,3} - k_n x_{1,2}) = 0$ whereas the solution is reduced to

$$\sin(k_n x_{1,2}) = \sin(k_n x_{1,3}) = 0$$

which is obtained for

$$\begin{array}{rclcrcl} k_n x_{1,2} & = & m\pi & \wedge & k_n x_{1,3} & = & l\pi \end{array} \qquad \begin{array}{rcl} m & = & 1, 2, 3, ... \\ l & = & m+1, m+2, ... \end{array}$$

$$\begin{array}{rclcrcl} k_n x_{1,2} & = & m\pi & \wedge & k_n x_{1,3} & = & l\pi \\ x_{1,2} & = & \frac{m}{2} L_n & \wedge & x_{1,3} & = & \frac{l}{2} L_n = \frac{l}{m} x_{1,2} \end{array}$$

as by definition $0 < x_{1,2} < x_{1,3}$. Mansard & Funke (1980) suggest that

$$\begin{array}{rcl} x_{1,2} & = & \frac{L_n}{10} \\ \frac{L_n}{6} & < & x_{1,3} < \frac{L_n}{3} \\ x_{1,3} & \neq & \frac{L_n}{5} \\ x_{1,3} & \neq & \frac{3L_n}{10} \end{array}$$

2.3 Zelt & Skjelbreia's method

Zelt and Skjelbreia (1992) introduced a method based on the same principles as the previously described methods.

This method applies for an arbitrary number p of probes and further introduces a weighting of the measurements from the gauges. The latter facility takes into account the spacing between each pair of wave gauges. However determining the weighting coefficients is a rather subjective process and requires some additional work.

For $p = 2$ the method corresponds to the method by Goda & Suzuki. For $p = 3$ and a uniform weighting, the method corresponds to the method by Mansard & Funke.

Once again the wave elevation is considered as a sum of incident and reflected waves, i.e. for the pth probe at position x_p utilising eq. (??)

$$\begin{aligned}\eta_p = \eta(x_p,t) = & \textstyle\sum_{n=1}^{N} a_{I,n}\cos(k_n x_p - \omega_n t + \Phi_{I,n}) + \\ & \textstyle\sum_{n=1}^{N} a_{R,n}\cos(k_n x_p + \omega_n t + \Phi_{R,n})\end{aligned}$$

or by use of eq. (??)

$$\begin{aligned}\eta_p &= \sum_{n=1}^{N} a_{I,n}\cos(k_n(x_1 + x_{1,p}) - \omega_n t + \Phi_n) \\ &\quad + \sum_{n=1}^{N} a_{R,n}\cos(k_n(x_1 + 2x_{R,1} - x_{1,p}) + \omega_n t + \Phi_n + \theta_s) + \Omega_p(t)\end{aligned}$$

As in the method of Mansard & Funke (1980) this is by use of the Fourier coefficients rearranged to eq. (??) which has the general form

$$\varepsilon_{p,n} = Z_{I,n}\exp(ik_n x_{1,p}) + Z_{R,n}\exp(-ik_n x_{1,p}) - (A_{p,n} + iB_{p,n}) \tag{2.27}$$

That is, if the estimated coefficients are correct there will be no error, i.e. $\varepsilon_{p,n} = 0$. A function is chosen to be a weighted sum of the squares of $\varepsilon_{p,n}$, i.e.

$$E_n = \sum_{p=1}^{P} W_{p,n}\varepsilon_{p,n}\varepsilon^*_{p,n} \tag{2.28}$$

where $W_{p,n} > 0$ are weighting coefficients.

It is assumed that the minimum of eq. (??) occurs where the partial derivatives with regard to $Z_{I,n}$ and $Z_{R,n}$ are zero. Thus

$$\sum_{p=1}^{P} W_{p,n}\varepsilon_{p,n}\exp(\pm ik_n x_{1,p}) = 0 \tag{2.29}$$

Inserting eq. (??) into eq. (??) gives:

$$\begin{aligned}\textstyle\sum_{p=1}^{P} W_{p,n} Z_{I,n}\exp(i2k_n x_{1,p}) + \sum_{p=1}^{P} W_{p,n} Z_{R,n} &= \\ \textstyle\sum_{p=1}^{P} W_{p,n}(A_{p,n} + iB_{p,n})\exp(ik_n x_{1,p}) & \\ \textstyle\sum_{p=1}^{P} W_{p,n} Z_{I,n} + \sum_{p=1}^{P} W_{p,n} Z_{R,n}\exp(-i2k_n x_{1,p}) &= \\ \textstyle\sum_{p=1}^{P} W_{p,n}(A_{p,n} + iB_{p,n})\exp(-ik_n x_{1,p}) &\end{aligned} \tag{2.30}$$

This, eq. (??) can be solved with regard to $Z_{I,n}$ and $Z_{R,n}$ as a linear system of equations. Simply isolate $a_{I,n}$ and $a_{R,n}$ in both equations in eq. (??) and substitute into the other equation respectively. Hence the following solution is obtained

$$\begin{aligned}Z_{I,n} &= \tfrac{1}{D_n}\Big(S_n \textstyle\sum_{p=1}^{P} W_{p,n}(A_{p,n} + iB_{p,n})\exp(-ik_n x_{1,p}) \\ &\quad - \textstyle\sum_{p=1}^{P} W_{p,n}(A_{p,n} + iB_{p,n})\exp(ik_n x_{1,p})\sum_{q=1}^{P} W_{q,n}\exp(-i2k_n x_{1,q})\Big) \\ Z_{R,n} &= \tfrac{1}{D_n}\Big(S_n \textstyle\sum_{p=1}^{P} W_{p,n}(A_{p,n} + iB_{p,n})\exp(ik_n x_{1,p}) \\ &\quad - \textstyle\sum_{p=1}^{P} W_{p,n}(A_{p,n} + iB_{p,n})\exp(-ik_n x_{1,p})\sum_{q=1}^{P} W_{q,n}\exp(i2k_n x_{1,q})\Big)\end{aligned}$$

(2.31)

where

$$D_n = S_n^2 - \sum_{p=1}^{P} W_{p,n} \exp(i2k_n x_{1,p}) \sum_{q=1}^{P} W_{q,n} \exp(-i2k_n x_{1,q})$$

$$S_n = \sum_{p=1}^{P} W_{p,n}$$

It is seen from eq. (??) that $D_n = 0$ will lead to singularity whereas it should be avoided that $x_{q,p} = 0$ for all p and q where $p > q$.

2.4 Frigaard & Brorsen's method

Contrary to all the previously described methods, the method by Frigaard & Brorsen (1993) works in *time domain.*

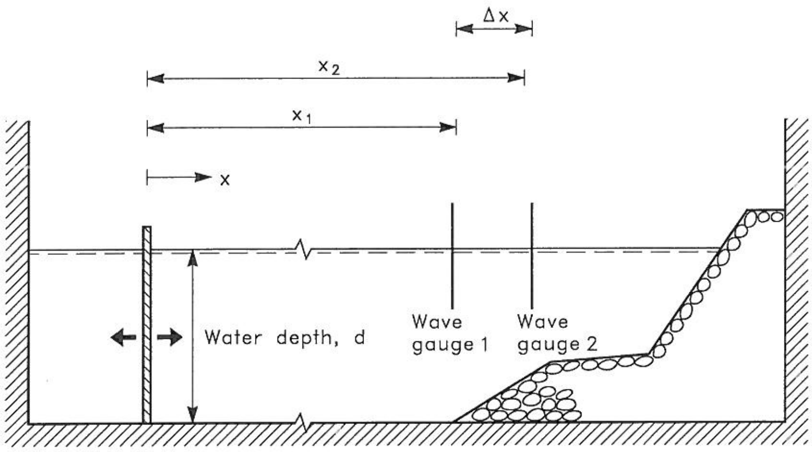

Figure 2.3: Wave channel with piston-type wave generator.

To illustrate the principle of the SIRW-method the set-up shown in fig. ?? will be considered. The surface elevation $\eta(x,t)$ at a distance x from the wave generator may be written as the sum of the incident and reflected waves. The incident wave propagating away from the wave generator, and the reflected wave propagating towards the wave generator. Even though the method works for irregular waves it will be demonstrated in the following pages for the case of monochromatic waves.

$$\begin{aligned}\eta(x,t) &= \eta_I(x,t) + \eta_R(x,t) \\ &= a_I \cos(\omega t - kx + \Phi_I) + a_R \cos(\omega t + kx + \Phi_R)\end{aligned} \tag{2.32}$$

where

ω	is cyclic frequency
$a = a(\omega)$	is wave amplitude
$k = k(\omega)$	is wave number
$\Phi = \Phi(\omega)$	is phase

and indices I and R denote incident and reflected, respectively.

At the two wave gauges we have:

$$\eta(x_1,t) = a_I \cos(\omega t - kx_1 + \Phi_I) + a_R \cos(\omega t + kx_1 + \Phi_R) \tag{2.33}$$

$$\begin{aligned}\eta(x_2,t) &= a_I \cos(\omega t - kx_2 + \Phi_I) + a_R \cos(\omega t + kx_2 + \Phi_R) \\ &= a_I \cos(\omega t - kx_1 - k\Delta x + \Phi_I) + \\ &\quad a_R cos(\omega t + kx_1 + k\Delta x + \phi_R)\end{aligned} \tag{2.34}$$

where $x_2 = x_1 + \Delta x$ has been substituted into eq. (??).

It is seen that the incident wave is phase shifted $\Delta\Phi = k\Delta x$ from signal $\eta(x_1,t)$ to signal $\eta(x_2,t)$, and the reflected wave is phase shifted $\Delta\Phi = -k\Delta x$ due to opposite travel directions. These phase shifts are called the physical phase shifts and are denoted Φ_I^{phys} and Φ_R^{phys}, respectively.

The idea in the following manipulations of the elevation signals is to phaseshift the signals from the two wave gauges in such ways that the incident parts of the wave signals are in phase while the reflected parts of the signals are in mutual opposite phase. In this case the sum of the two manipulated signals is proportional to and in phase with the incident wave signal.

An amplification C and a theoretical phase shift Φ^{theo} are introduced into the expressions for $\eta(x,t)$. The modified signal is denoted η^*. For the i'th wave gauge signal the modified signal is defined as:

$$\begin{aligned}\eta^*(x_i,t) &= Ca_I \cos(\omega t - kx_i + \Phi_I + \Phi_i^{theo}) + \\ &\quad Ca_R \cos(\omega t + kx_i + \Phi_R + \Phi_i^{theo})\end{aligned} \tag{2.35}$$

This gives at wave gauges 1 and 2:

$$\begin{aligned}
\eta^*(x_1,t) &= Ca_I\cos(\omega t - kx_1 + \Phi_I + \Phi_1^{theo}) + \\
&\quad Ca_R\cos(\omega t + kx_1 + \Phi_R + \Phi_1^{theo})
\end{aligned} \tag{2.36}$$

$$\begin{aligned}
\eta^*(x_2,t) &= Ca_I\cos(\omega t - kx_2 + \Phi_I + \Phi_2^{theo}) + \\
&\quad Ca_R\cos(\omega t + kx_2 + \Phi_R + \Phi_2^{theo}) \\
&= Ca_I\cos(\omega t - kx_1 - k\Delta x + \Phi_I + \Phi_2^{theo}) + \\
&\quad Ca_R\cos(\omega t + kx_1 + k\Delta x + \Phi_R + \Phi_2^{theo})
\end{aligned} \tag{2.37}$$

The sum of $\eta^*(x_1,t)$ and $\eta^*(x_2,t)$, which is denoted $\eta^{calc}(t)$, gives:

$$\begin{aligned}
\eta^{calc}(t) &= \eta^*(x_1,t) + \eta^*(x_2,t) \\
&= Ca_I\cos(\omega t - kx_1 + \Phi_I + \Phi_1^{theo}) + \\
&\quad Ca_R\cos(\omega t + kx_1 + \Phi_R + \Phi_1^{theo}) + \\
&\quad Ca_I\cos(\omega t - kx_1 - k\Delta x + \Phi_I + \Phi_2^{theo}) + \\
&\quad Ca_R\cos(\omega t + kx_1 + k\Delta x + \Phi_R + \Phi_2^{theo}) \\
&= 2Ca_I\cos(0.5(-k\Delta x - \Phi_1^{theo} + \Phi_2^{theo})) \\
&\quad \cos(\omega t - kx_1 + \Phi_I + 0.5(-k\Delta x + \Phi_1^{theo} + \Phi_2^{theo})) + \\
&\quad 2Ca_R\cos(0.5(-k\Delta x + \Phi_1^{theo} - \Phi_2^{theo})) \\
&\quad \cos(\omega t + kx_1 + \Phi_R + 0.5(k\Delta x + \Phi_1^{theo} + \Phi_2^{theo}))
\end{aligned} \tag{2.38}$$

It is seen that $\eta^{calc}(t)$ and $\eta_I(x_1,t) = a_I\cos(\omega t - kx_1 + \Phi_I)$ are identical signals in case:

$$2C\cos(0.5(-k\Delta x - \Phi_1^{theo} + \Phi_2^{theo})) = 1 \tag{2.39}$$

$$0.5(-k\Delta x + \Phi_1^{theo} + \Phi_2^{theo}) = n\cdot 2\pi \qquad n \in (0, \pm 1, \pm 2, ..) \tag{2.40}$$

$$0.5(-k\Delta x + \Phi_1^{theo} - \Phi_2^{theo}) = \frac{\pi}{2} + m\cdot\pi \qquad m \in (0, \pm 1, \pm 2, ..) \tag{2.41}$$

Solving eq. (??) - eq. (??) with respect to $\Phi_1^{theo}, \Phi_2^{theo}$ and C gives eq. (??) - eq. (??). n and m can still be chosen arbitrarily. Thus:

$$\Phi_1^{theo} = k\Delta x + \pi/2 + m\pi + n2\pi \quad (2.42)$$

$$\Phi_2^{theo} = -\pi/2 - m\pi + n2\pi \quad (2.43)$$

$$C = \frac{1}{2\cos(-k\Delta x - \pi/2 - m\pi)} \quad (2.44)$$

All the previous considerations and calculations were done in order to find an amplification and a phaseshift for each of the two elevation signals η_1 and η_2.

Eq. (??) - eq. (??) give the result of our efforts, i.e. $\eta_I(x_1,t) = \eta^{calc}(t)$. Remembering that $\Phi_1^{theo} = \Phi_1^{theo}(\omega), \Phi_2^{theo} = \Phi_2^{theo}(\omega)$ and $C = C(\omega)$ it is seen that the aim is already reached in *frequency domain*. However, the implementation of the principle will be done in *time domain* using digital filters.

It is seen that singularities may occur. The consequences and the handling of the singularities will be treated later on. Here it should just be mentioned that one way to bypass the singularities is to use a velocity meter instead of one of the two wave gauges.

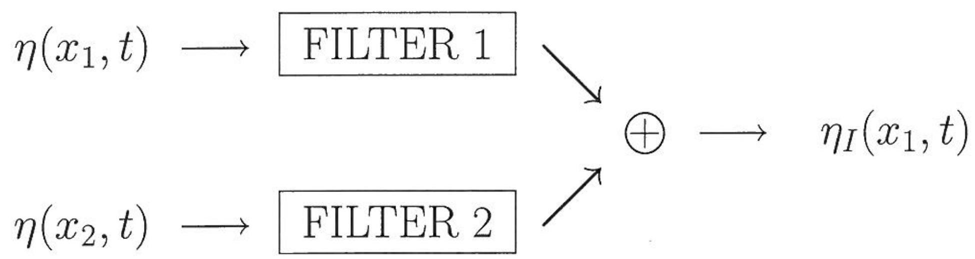

Figure 2.4: Flow diagram for signals in the method.

The purposes of the filters shown in fig. ?? are a frequency dependent amplification and a frequency dependent phaseshift on each of the two elevation signals.

Taking $n = 0$ and $m = 0$ the frequency response functions $H_1(\omega)$ for filter 1 and $H_2(\omega)$ for filter 2 calculated due to eq. (??) - eq. (??) are given on next page in complex notation:

$$
\begin{aligned}
Re\{H_1(\omega)\} &= \frac{1}{2\cos(-k\Delta x - \pi/2)} \cdot \cos(k\Delta x + \pi/2) \\
Im\{H_1(\omega)\} &= \frac{1}{2\cos(-k\Delta x - \pi/2)} \cdot \sin(k\Delta x + \pi/2) \qquad (2.45) \\
Re\{H_2(\omega)\} &= \frac{1}{2\cos(-k\Delta x - \pi/2)} \cdot \cos(-\pi/2) \\
Im\{H_2(\omega)\} &= \frac{1}{2\cos(-k\Delta x - \pi/2)} \cdot \sin(-\pi/2) \qquad (2.46)
\end{aligned}
$$

Based on eq. (??) and (??) it is straight forward to design the time domain filters.

2.5 Oblique waves

In the case of oblique waves, i.e. 2D-waves travelling along a line not perpendicular to the reflecting structure, the previous four methods all can be applied by assuming that the waves can be decomposed into two vectorial components, being respectively $\perp$ and $||$ to the structure.

$$
\begin{aligned}
\eta(\boldsymbol{x}, t) = & \textstyle\sum_{n=1}^{N} a_{I,n} \cos(\boldsymbol{k}_n \boldsymbol{x} - \omega_n t + \Phi_{I,n}) + \\
& \textstyle\sum_{n=1}^{N} a_{R,n} \cos(\boldsymbol{k}_n \boldsymbol{x} + \omega_n t + \Phi_{R,n}) \qquad (2.47)
\end{aligned}
$$

where η is the surface elevation relative to MWL
$\boldsymbol{x}$ is the vectorial position of the wave gauge
t is time
a_n is the amplitude
$\boldsymbol{k}_n$ is the vectorial wavenumber
ω_n is the angular frequency of the waves
Φ_n is the phase

Omitting index n and rewriting eq. (??) in cartesian coordinates:

$$\begin{aligned}\eta(x,y,t) = \; & a_I \cos(\; kx\cos(\theta_I) + ky\sin(\theta_I) - \omega t + \Phi_I) + \\ & a_R \cos(\; kx\cos(\theta_R) + ky\sin(\theta_R) + \omega t + \Phi_R)\end{aligned} \tag{2.48}$$

where x, y is the vectorial position of the wave gauge
k is the wavenumber
θ is direction of wave

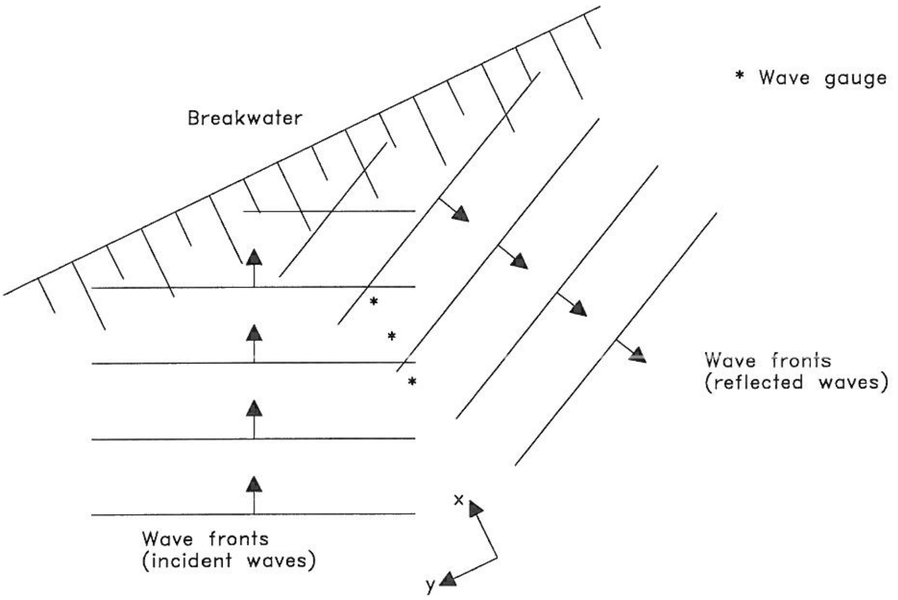

Figure 2.5: Placement of wave gauges in front of structure.

One example of solving the problem is to place all wave gauges on a line with y-coordinate $\equiv 0$ as shown in fig. ??. Eq. (??) will then simplify to:

$$\begin{aligned}\eta(x,0,t) = \; & a_I \cos(kx\cos(\theta_I) - \omega t + \Phi_I) + \\ & a_R \cos(kx\cos(\theta_R) + \omega t + \Phi_R)\end{aligned} \tag{2.49}$$

It is seen that if the angle of the incident wave is know, and the angle of the reflected wave is calculated using Snells law (incident angle = reflected angle) eq. (??) is very similar to the original expression for the wave elevation, eq(??).

The easiest way to solve the problem is to place the wave gauges on a line perpendicular to the reflecting structure.

Please notice that near the structure *edge-waves* will exist. These waves travel along the structure and can destroy the calculations because they are not included in the modelling of the waves.

2.6 References

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3 Three-dimensional methods

3.1 Introduction

A short crested wave field is normally referred to as a three-dimensional wavefield. This implies introduction of an additional parameter namely the direction of travel. A two dimensional wavefield is commonly described in the frequency domain by use of the wavespectrum, i.e. the autospectrum of the waveelevation process. Now in the three dimensional case a directional spreading function depending on frequency and direction of travel is introduced. The combination of the wavespectrum and the spreading function is the directional wavespectrum. Having determined the directional wavespectrum the wavefield is fully described in the frequency domain.

For a two dimensional seastate the elevation was assumed to be a summation of a number of wavelets as stated in e.g. eq. (??). In a three dimensional case the corresponding expression is

$$\eta(\boldsymbol{x}, t) = \sum_{n=1}^{N} \sum_{m=1}^{M} a_{mn} \cos(\boldsymbol{k}_{mn}\boldsymbol{x} - \omega_n t + \Phi_{mn}) \tag{3.1}$$

where $\boldsymbol{k}$ is the wavenumber vector. Eq. (??) implies that for each pair of discrete values of frequency and direction of travel there exist a long crested wave propagating with these properties.

The amplitude a_{mn} can alternatively be described by use of the wavespectrum as the variance of a sinusoidal wave is half the amplitude squared. Hence

$$\begin{aligned} \frac{1}{2}a_{mn}^2 &= S(\omega_m, \theta_n)\Delta\theta\Delta\omega \\ a_{mn} &= \sqrt{2S(\omega_m, \theta_n)\Delta\theta\Delta\omega} \end{aligned} \tag{3.2}$$

Inserting eq. (??) into eq. (??) yields

$$\eta(\boldsymbol{x},t) = \sum_{n=1}^{N}\sum_{m=1}^{M}\sqrt{2S(\omega_m,\theta_n)\Delta\theta\Delta\omega}\cos(\boldsymbol{k}_{mn}\boldsymbol{x} - \omega_n t + \Phi_{mn}) \tag{3.3}$$

Letting $\Delta\theta \rightarrow d\theta$ and $\Delta\omega \rightarrow d\omega$ eq. (??) turns into a double integral

$$\eta(\boldsymbol{x},t) = \int_0^\infty \int_{-\pi}^{\pi} \cos(\boldsymbol{k}\boldsymbol{x} - \omega t + \Phi(\omega,\theta))\sqrt{2S(\omega,\theta)d\theta d\omega} \tag{3.4}$$

The aim of the following is to establish an expression which relates known and measured numbers to the directional wavespectrum or the directional spreading function. These latter functions are related through

$$S(\omega,\theta) \quad = \quad H(\omega,\theta)S(\omega) \tag{3.5}$$

Two wave gauges are considered. These are positioned as shown in fig. ??.

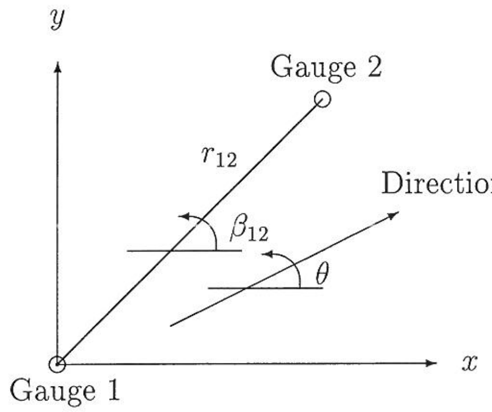

Figure 3.6: Definition of geometric parameters.

The elevations at gauge 1 and gauge 2 respectively are

$$\begin{aligned}
\eta_1 = \eta(\boldsymbol{x}_1,t) &= \int_0^\infty \int_{-\pi}^{\pi} \cos(\boldsymbol{k}\boldsymbol{x}_1 - \omega t + \Phi(\omega,\theta))\sqrt{2S(\omega,\theta)d\theta d\omega} \\
\eta_2 = \eta(\boldsymbol{x}_2,t) &= \int_0^\infty \int_{-\pi}^{\pi} \cos(\boldsymbol{k}\boldsymbol{x}_2 - \omega t + \Phi(\omega,\theta))\sqrt{2S(\omega,\theta)d\theta d\omega} \\
&= \int_0^\infty \int_{-\pi}^{\pi} \cos(\boldsymbol{k}\boldsymbol{x}_1 - \omega t + kr_{12}\cos(\theta - \beta_{12}) + \Phi(\omega,\theta)) \cdot \\
&\quad \sqrt{2S(\omega,\theta)d\theta d\omega}
\end{aligned}$$

The cross-correlationfunction is obtained as

$$R_{\eta_1\eta_2}(\tau) \quad = \quad \frac{1}{T}\int_0^T \eta_1(t)\eta_2(t+\tau)dt \tag{3.6}$$

Here it is necessary to specify the time argument, thus $\eta_1(t) = \eta_1$ and $\eta_2(t) = \eta_2$. Inserting into eq. (??) yields

$$\begin{aligned} R_{\eta_1\eta_2}(\tau) &= \frac{1}{T}\int_0^T \int_0^\infty \int_{-\pi}^{\pi} \cos(\boldsymbol{k}\boldsymbol{x}_1 - \omega t + \Phi(\omega,\theta)) \cdot \\ &\quad \cos(\boldsymbol{k}\boldsymbol{x}_2 - \omega(t+\tau) + \Phi(\omega,\theta)) 2S(\omega,\theta) d\theta d\omega dt \end{aligned} \tag{3.7}$$

Applying the trigonometric relations

$$\begin{aligned} 2\cos\alpha\cos\beta &= \cos(\alpha-\beta) + \cos(\alpha+\beta) \\ \cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \end{aligned}$$

leads to

$$\begin{aligned} R_{\eta_1\eta_2}(\tau) &= \frac{1}{T}\int_0^T \int_0^\infty \int_{-\pi}^{\pi} [\cos(kr_{12}\cos(\theta-\beta_{12}) - \omega t) \\ &\quad + \cos(2\boldsymbol{k}\boldsymbol{x}_1 + kr_{12}\cos(\theta-\beta_{12}) - 2\omega t - \omega\tau + 2\Phi(\omega,\theta))] \cdot \\ &\quad S(\omega,\theta) d\theta d\omega dt \\ &= \int_0^\infty \int_{-\pi}^{\pi} \cos(kr_{12}\cos(\theta-\beta_{12}) - \omega\tau) S(\omega,\theta) d\theta d\omega \\ &= \int_0^\infty \int_{-\pi}^{\pi} [\cos(\omega\tau)\cos(kr_{12}\cos(\theta-\beta_{12})) \\ &\quad + \sin(\omega\tau)\sin(kr_{12}\cos(\theta-\beta_{12}))] S(\omega,\theta) d\theta d\omega \\ &= \int_0^\infty c_{12}(\omega)\cos(\omega\tau) d\omega + \int_0^\infty q_{12}(\omega)\sin(\omega\tau) d\omega \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} c_{12}(\omega) &= \int_{-\pi}^{\pi} S(\omega,\theta)\cos(kr_{12}\cos(\theta-\beta_{12})) d\theta \\ q_{12}(\omega) &= \int_{-\pi}^{\pi} S(\omega,\theta)\sin(kr_{12}\cos(\theta-\beta_{12})) d\theta \end{aligned}$$

is the co-spectrum and the quad-spectrum respectively. Thus

$$S_{\eta_1\eta_2}(\omega) = c_{12}(\omega) - iq_{12}(\omega) \tag{3.9}$$

Inserting the co- and quad-spectrum into eq. (??) yields

$$\begin{aligned} S^*_{\eta_1\eta_2}(\omega) &= \int_{-\pi}^{\pi} S(\omega,\theta)\exp(ikr_{12}\cos(\theta-\beta_{12})) d\theta \\ \frac{S^*_{\eta_1\eta_2}(\omega)}{S_{\eta\eta}(\omega)} &= \int_{-\pi}^{\pi} H(\omega,\theta)\exp(kr_{12}\cos(\theta-\beta_{12})) d\theta \end{aligned} \tag{3.10}$$

Eq. (??) relates data, which can be computed from wave elevation measurements, to the unknown directional spreading function. The geometry is represented by r_{12} and β_{12}.

Eq. (??) is the basic relation that was in demand. It cannot be solved analytically. Therefore methods have been proposed in order to make a reliable estimate.

It is common to all methods that they initially suggest some shape of the directional spreading function. That may either be a parameterized analytic expression or a number of discrete values making a step curve. This initial assumption is very important to the ability of the particular method. The next step is to fit the unknown parameters to the measured data. The reliability of the fit depends on the reliability of the measured data but also on the number of coefficients to fit, i.e. the more coefficients the less reliability. This is a problem as a high resolution requires many coefficients.

3.2 The Maximum Likelihood Method

In the Maximum Likelihood Method, MLM, the directional spectrum is calculated from minimizing the errors between measured wave data (cross-correlations) and fitted directional spectrum.

This is achieved by use of the Maximum Likelihood technique, which has named the method.

In it most simple form MLM applies for simultaneous wave elevations, but by use of transfer functions it can readily be extended to apply for various measurements. The MLM was originally presented by Capon (1969), but has since been modified by several authors. Especially the papers by Davis and Regier (1977) and Isobe et al. (1984) are of importance.

As an initial assumption the directional spectrum is expressed as a linear combination of the cross-spectra. The linear wave theory is assumed valid as well, i.e.

$$\begin{aligned} k &= 2\pi/L \\ L &= L_0 \tanh(kd) \end{aligned}$$

Hence the estimated directional spectrum, $\tilde{S}(\omega,\theta)$, can be written as

$$\tilde{S}(\omega,\theta) = \sum_{m=1}^{N}\sum_{n=1}^{N} \alpha_{mn}(\omega,\theta) S_{mn}(\omega) \tag{3.11}$$

Then using

$$S_{mn}(\omega) = \int_0^{2\pi} \exp(-i\boldsymbol{k}(\boldsymbol{x}_n - \boldsymbol{x}_m)) S(\omega,\theta) d\theta \tag{3.12}$$

eq. (??) leads to

$$\tilde{S}(\omega,\theta) = \sum_{m=1}^{N}\sum_{n=1}^{N}\alpha_{mn}(\omega,\theta)\int_0^{2\pi}\exp(-i\boldsymbol{k}'(\boldsymbol{x}_n-\boldsymbol{x}_m))S(\omega,\theta')d\theta' \qquad (3.13)$$

where $\boldsymbol{k}' = \left\{ \begin{array}{c} k\cos\theta' \\ k\sin\theta' \end{array} \right\}$.

Eq. (??) is expressed as

$$\tilde{S}(\omega,\theta) = \int_0^{2\pi} w(\omega,\theta,\theta')S(\omega,\theta')d\theta' \qquad (3.14)$$

where

$$w(\omega,\theta,\theta') = \sum_{m=1}^{N}\sum_{n=1}^{N}\alpha_{mn}(\omega,\theta)\exp(-i\boldsymbol{k}'(\boldsymbol{x}_n-\boldsymbol{x}_m)) \qquad (3.15)$$

From eq. (??) it is seen that the estimated directional spectrum is a convolution of the true directional spectrum and the window function $w(\omega,\theta,\theta')$ given by eq. (??).

It is assumed that the coefficients $\alpha_{mn}(\omega,\theta)$ can be expressed as

$$\alpha_{mn}(\omega,\theta) = \gamma_m(\omega,\theta)\gamma_n^*(\omega,\theta) \qquad (3.16)$$

Inserting this into eq. (??) and eq. (??) yields

$$\tilde{S}(\omega,\theta) = \sum_{m=1}^{N}\sum_{n=1}^{N}\gamma_m(\omega,\theta)\gamma_n^*(\omega,\theta)S_{mn}(\omega) \qquad (3.17)$$

$$\begin{aligned} w(\omega,\theta,\theta') &= \sum_{m=1}^{N}\sum_{n=1}^{N}\gamma_m(\omega,\theta)\gamma_n^*(\omega,\theta)\exp(-i\boldsymbol{k}'(\boldsymbol{x}_n-\boldsymbol{x}_m)) \\ &= \sum_{m=1}^{N}\sum_{n=1}^{N}\gamma_m(\omega,\theta)\gamma_n^*(\omega,\theta)\exp(i\boldsymbol{k}'\boldsymbol{x}_m)\exp(-i\boldsymbol{k}'\boldsymbol{x}_n) \\ &= \sum_{m=1}^{N}\gamma_m(\omega,\theta)\exp(i\boldsymbol{k}'\boldsymbol{x}_m)\sum_{n=1}^{N}\gamma_n^*(\omega,\theta)\exp(-i\boldsymbol{k}'\boldsymbol{x}_n) \\ &= \sum_{m=1}^{N}\gamma_m(\omega,\theta)\exp(i\boldsymbol{k}'\boldsymbol{x}_m)\sum_{n=1}^{N}(\gamma_n(\omega,\theta)\exp(i\boldsymbol{k}'\boldsymbol{x}_n))^* \\ &= |\sum_{m=1}^{N}\gamma_m(\omega,\theta)\exp(i\boldsymbol{k}'\boldsymbol{x}_m)|^2 \end{aligned} \qquad (3.18)$$

The window function is normalised by setting $w(\omega, \theta, \theta) = 1$. Having done this it is seen from eq. (??) that if the window function is Diracs delta function, then the estimated directional spectrum is equal to the true directional spectrum. As both the window function and the true directional spectrum is non-negative the aim is to minimise $\tilde{S}(\omega, \theta)$ as given in eq. (??) i.e.

$$\text{minimise}\left(\sum_{m=1}^{N}\sum_{n=1}^{N}\gamma_m(\omega,\theta)\gamma_n^*(\omega,\theta)S_{mn}(\omega)\right)$$

The same problem can be formulated as

$$\text{maximise}\left(\frac{w(\omega,\theta,\theta)}{\tilde{S}(\omega,\theta)} = \frac{\sum_{m=1}^{N}\sum_{n=1}^{N}\gamma_m(\omega,\theta)T_{mn}(\omega,\theta)\gamma_n^*(\omega,\theta)}{\sum_{m=1}^{N}\sum_{n=1}^{N}\gamma_m(\omega,\theta)S_{mn}(\omega,\theta)\gamma_n^*(\omega,\theta)}\right) \tag{3.19}$$

where a matrix $\boldsymbol{T}$ has been introduced as

$$\begin{aligned} T_{mn}(\omega,\theta) &= \exp(i\boldsymbol{k}\boldsymbol{x}_m)\exp(-i\boldsymbol{k}\boldsymbol{x}_n) &(3.20)\\ &= \gamma_{om}^*(\omega,\theta)\gamma_{on}(\omega,\theta) &(3.21)\end{aligned}$$

The maximum value of eq. (??) is equal to the maximum eigenvalue of the matrix $\boldsymbol{S}^{-1}\boldsymbol{T}$. Eq. (??) is then the Rayleigh quotient of $\boldsymbol{\gamma}^T(\boldsymbol{T} - \lambda\boldsymbol{S})\boldsymbol{\gamma}^* = 0$ whereas

$$\text{maximum}\left(\frac{\boldsymbol{\gamma}^T\boldsymbol{T}\boldsymbol{\gamma}^*}{\boldsymbol{\gamma}^T\boldsymbol{S}\boldsymbol{\gamma}^*}\right) = \lambda_{max} \tag{3.22}$$

where λ_{max} is the maximum eigenvalue. Thus by multiplication of $\gamma^{T^{-1}}$ and $\boldsymbol{S}^{-1}$ and substitution of $\boldsymbol{T}$

$$\begin{aligned} \boldsymbol{\gamma}^T\boldsymbol{T}\boldsymbol{\gamma}^* &= \lambda_{max}\boldsymbol{\gamma}^T\boldsymbol{S}\boldsymbol{\gamma}^*\\ \boldsymbol{T}\boldsymbol{\gamma}^* &= \lambda_{max}\boldsymbol{S}\boldsymbol{\gamma}^*\\ \boldsymbol{S}^{-1}\boldsymbol{T}\boldsymbol{\gamma}^* &= \lambda_{max}\boldsymbol{\gamma}^*\\ \boldsymbol{S}^{-1}\boldsymbol{\gamma}_o^*\boldsymbol{\gamma}_o^T\boldsymbol{\gamma}^* &= \lambda_{max}\boldsymbol{\gamma}^*\end{aligned}$$

Further by premultiplication by $\boldsymbol{\gamma}_o^T$ yields

$$\boldsymbol{\gamma}_o^T\boldsymbol{S}^{-1}\boldsymbol{\gamma}_o^*\boldsymbol{\gamma}_o^T\boldsymbol{\gamma}^* = \lambda_{max}\boldsymbol{\gamma}_o^T\boldsymbol{\gamma}^* \tag{3.23}$$

Thus

$$\boldsymbol{\gamma}_o^T\boldsymbol{S}^{-1}\boldsymbol{\gamma}_o^* = \lambda_{max} \tag{3.24}$$

It is given from eq. (??) that

$$\tilde{S}(\omega,\theta) \propto 1/\lambda_{max}$$

Hence introducing a proportionality factor κ the directional spectrum can be estimated as

$$\tilde{S}(\omega,\theta) \quad = \quad \kappa\left(\gamma_o^T \boldsymbol{S}^{-1}\gamma_o^*\right)^{-1} \tag{3.25}$$

The terms in the RHS of eq. (??) are all known when a sample has been carried out. $\boldsymbol{S}$ is the cross-spectrum matrix and is calculated from the timeseries. γ_o is dependent only on the wavenumber vector and the geometry. The factor κ is used in order to achieve the correct variance (i.e. the measured variance). The MLM is for the time being considered as one of the best methods for estimation of directional wavespectra. It is however not reliable when it is applied to a sea state which include reflection. This is due to the presence of waves travelling with exactly the same frequency. This problem arises in most methods.

MLM has been derived alternatively where a parametric form of the spreading functions are presumed. Hiromune et al. (1992) have done this utilising Mitsuyasus spreading function. Further the spreading function has the option to imply a reflected wave system which in spreading has the same form as the incoming system but reduced in energy. This method assumes that the waves are reflected along a structure that has a straight front.

3.3 MLM utilising standard spectra

The purpose of the present section is to present the Maximum Likelihood method for estimating directional spectra utilising standard spectra. The presentation is based on Isobe & Kondo (1984), Isobe (1990), Yokoki, Isobe & Watanabe (1992) and Christensen & Sørensen (1994). The directional spectrum is given in a standard form in terms of some unknown parameters to be estimated from measured data. In the present section only surface elevations measurements are treated.

The starting point is M surface elevations, $\eta(\mathbf{x},t)$, measured at M different locations $\mathbf{x}$ at time t. The total elevation processes $\eta_p\left(\boldsymbol{x}_p,t\right)$, $p=1,2,...,M$, are modelled as stochastic processes. The processes are assumed to be joint stationary, ergodic and Gaussian distributed. The mean value functions $\mu_{\eta_p}(t)$, $p=1,2,\ldots,M$, are assumed to be equal to 0. The M time series can be written as a Fourier-sum as

$$\eta_p(\boldsymbol{x},t) = \sum_{l=1}^{N}\left(A_{p,l}\cos\omega_l t + B_{p,l}\sin\omega_l t\right)\ ,\ p=1,2,...,M \tag{3.26}$$

where $\omega_l = l\frac{2\pi}{T}$ and T is the length of the time series. The coefficients $A_{p,l}$ and $B_{p,l}$ are given as the stochastic integrals

$$A_{p,l} \quad = \quad \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \eta_p\left(\boldsymbol{x}_p,t\right)\cos\omega_l t\,dt \qquad l=1,2,\ldots,N\ p=1,2,\ldots,M \tag{3.27}$$

$$B_{p,l} = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \eta_p(\boldsymbol{x}_p, t)\sin\omega_l t\, dt \qquad l = 1, 2, \ldots N\ p = 1, 2, \ldots, M \tag{3.28}$$

In eq. (??) the term corresponding to $l = 0$ has been omitted as it equals 0.

From eq. (??) and eq. (??) it is seen that all the coefficients $A_{p,l}$ and $B_{p,l}$ are joint Gaussian distributed stochastic variables.

In the following only the l'th components are considered. The primary aim is to determine the joint distribution of the coefficients $A_{p,l}$ and $B_{p,l}$ $p = 1, 2, \ldots, M$, at the frequency w_l.

The reason for limiting the analysis to a single frequency at a time is that the unknown parameters in $S(\omega, \theta)$ to be estimated may be frequency dependent.

The coefficients are expressed as a vector

$$\begin{aligned} \boldsymbol{A}^T &= [A_1\ A_2\ \ldots\ A_M\ B_1\ B_2\ \ldots\ B_M] \\ &= [A_1\ \ldots\ A_M\ A_{M+1}\ \ldots\ A_{2M}] \end{aligned} \tag{3.29}$$

As the coefficients are joint Gaussian distributed the frequency distribution function is given in terms of the mean value function vector, $E[\boldsymbol{A}]$, and the cross covariance function matrix $\boldsymbol{\kappa}_{\boldsymbol{A}\boldsymbol{A}^T} = E\left[\boldsymbol{A}\,\boldsymbol{A}^T\right]$. The cross covariance function matrix of size $2M \times 2M$ is symmetric and of the form

$$\boldsymbol{\kappa}_{\boldsymbol{A}\boldsymbol{A}^T} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{E} \\ \boldsymbol{E}^T & \boldsymbol{D} \end{bmatrix} \tag{3.30}$$

where the submatrices $\boldsymbol{B}$, $\boldsymbol{E}$ and $\boldsymbol{D}$ are $M \times M$ matrices.

From eq. (??) and eq. (??) the following results are found

$$E[A_{p,l}] = \frac{2}{T}\int_{-\frac{2}{T}}^{\frac{2}{T}} E[\eta_p(\boldsymbol{x}_p, t)]\cos\omega_l t\, dt = 0 \quad , \quad p = 1, 2, \ldots, M \tag{3.31}$$

and

$$E[B_{p,l}] = \frac{2}{T}\int_{-\frac{2}{T}}^{\frac{2}{T}} E[\eta_p(\boldsymbol{x}_p, t)]\sin\omega_l t\, dt = 0 \quad , \quad p = 1, 2, \ldots, M \tag{3.32}$$

i.e. $E[\boldsymbol{A}] = \boldsymbol{0}$.

In the following the cross covariance function matrix $E\left[\boldsymbol{A}\,\boldsymbol{A}^T\right]$ will be determined. As an example the submatrix $\boldsymbol{B}$ in eq. (??) is considered.

$$\begin{aligned}
\kappa_{A_{m,l}A_{n,l}}\left(t_1,t_2\right) &= E\left[\frac{2}{T}\int_{-\frac{2}{T}}^{\frac{2}{T}} \eta_m\left(t_1\right)\,\cos\left(\omega_l t_1\right)dt_1 \cdot \frac{2}{T}\int_{-\frac{2}{T}}^{\frac{2}{T}} \eta_n\left(t_2\right)\,\cos\left(\omega_l t_2\right)dt_2\right] \\
&= \frac{4}{T^2}\int_{-\frac{2}{T}}^{\frac{2}{T}}\int_{-\frac{2}{T}}^{\frac{2}{T}} E\left[\eta_m\left(t_1\right)\eta_n\left(t_2\right)\right]\,\cos\left(\omega_l t_1\right)\cos\left(\omega_l t_2\right)\,dt_1 dt_2 \\
&= \frac{4}{T^2}\int_{-\frac{2}{T}}^{\frac{2}{T}}\int_{-\frac{2}{T}}^{\frac{2}{T}} \boldsymbol{\kappa}_{mn}\left(t_2-t_1\right)\cos\left(\omega_l t_1\right)\cos\left(\omega_l t_2\right)\,dt_1 dt_2 \qquad (3.33)
\end{aligned}$$

It is seen, that $\kappa_{A^m_{cj}A^n_{cj}}$ depends on the cross covariance between the elevation processes $\eta_m\left(t_1\right)$ and $\eta_n\left(t_2\right)$. The cross covariance matrix $\boldsymbol{\kappa}_{mn}(\tau)$ is related to the cross spectral density matrix $S_{mn}(\omega)$ in terms of the Wiener-Khinchine relation

$$\begin{aligned}
G_{mn}(\omega) &= 2S_{mn}(\omega) = 2\int_{-\infty}^{\infty} \boldsymbol{\kappa}_{mn}(\tau)\,e^{-i\omega\tau}\,d\tau \\
&= 2\int_{-\infty}^{\infty} \boldsymbol{\kappa}_{mn}(\tau)\,\cos\omega\tau\,d\tau - i\cdot 2\int_{-\infty}^{\infty} \boldsymbol{\kappa}_{mn}(\tau)\,\sin\omega\tau\,d\tau \\
&= C_{mn}(\omega) - i\,Q_{mn}(\omega) \qquad (3.34)
\end{aligned}$$

where $G_{mn}(\omega)$ is a one-sided spectrum.

$$C_{mn}(\omega) = 2\int_{-\infty}^{\infty} \boldsymbol{\kappa}_{mn}(\tau)\,\cos\omega\tau\,d\tau$$

and

$$Q_{mn}(\omega) = 2\int_{-\infty}^{\infty} \boldsymbol{\kappa}_{mn}(\tau)\,\sin\omega\tau\,d\tau$$

are the co-spectrum and the quad-spectrum, respectively.

Assuming the period T to be "very long" the following result is obtained from eq. (??)

$$\kappa_{A_{m,l}A_{n,l}}\left(t_1,t_2\right) = \kappa_{A_{m,l}A_{n,l}}(\tau) = \frac{1}{T}\,C_{mn}\left(\omega_l\right) = \frac{\Delta\omega}{2\pi}\,C_{mn}\left(\omega_l\right) = B_{mn} \qquad (3.35)$$

It is concluded that $\boldsymbol{B}$ in eq. (??) equals $\frac{\Delta\omega}{2\pi}$ multiplied by the co-spectrum of the elevation processes. Further $\boldsymbol{B}$ is time independent.

Using the same procedure the elements $\boldsymbol{E}$ and $\boldsymbol{D}$ of eq. (??) can be found and after some calculation the following result is obtained

$$\boldsymbol{\kappa}_{\boldsymbol{A}\boldsymbol{A}^T}\left(\omega_l\right) = \frac{\Delta\omega}{2\pi}\begin{bmatrix} \boldsymbol{C} & \boldsymbol{Q} \\ -\boldsymbol{Q} & \boldsymbol{C} \end{bmatrix} = \frac{\Delta\omega}{2\pi}\cdot\boldsymbol{\Omega}\left(\omega_l\right) \qquad (3.36)$$

i.e. the cross-covariance between the coefficients in eq. (??) at a given frequency ω_l is given in terms of the co- and quad-spectra of the elevation processes.

So far expressions have been established for the mean value vector (eq. (??) and eq. (??)) and the cross covariance matrix eq. (??) of the vector $\boldsymbol{A}$. These relations were established at a given frequency ω_l. Besides the frequency the wave pattern is also characterised by a direction of travel. The dependence of the frequency and the direction of travel is expressed in terms of the directional spectrum $S(\omega, \theta)$. The following relation exists between the one-sided autospectral density $G(\omega)$ and $S(\omega, \theta)$

$$G(\omega) = \int_0^{2\pi} S(\omega, \theta)\, d\theta \tag{3.37}$$

Furthermore, it can be shown that $S(\omega, \theta)$ is related to the cross spectral density matrix, $G_{mn}(\omega)$, of the elevation processes $\eta_m(\boldsymbol{x}_m, t)$ and $\eta_n(\boldsymbol{x}_n, t)$ as expressed in eq. (??)

$$\begin{aligned} G_{mn}(\omega) \quad &= \textstyle\int_0^{2\pi} S(\omega, \theta) \{ \exp(i\boldsymbol{k}\,(\boldsymbol{x}_m - \boldsymbol{x}_n) + r(\omega, \theta) \exp(i\boldsymbol{k}\,(\boldsymbol{x}_m - \boldsymbol{x}_n \boldsymbol{T})) \\ &+ r(\omega, \theta) \exp(i\boldsymbol{k}\,(\boldsymbol{x}_m \boldsymbol{T} - \boldsymbol{x}_n)) + r^2(\omega, \theta) \exp(i\boldsymbol{k}\,(\boldsymbol{x}_m - \boldsymbol{x}_n)\,\boldsymbol{T}) \} \end{aligned} \tag{3.38}$$

where $r(\omega, \theta)$ is the reflection coefficient. The modelling of reflected waves is described in Christensen & Sørensen (1994). $\boldsymbol{T}$ is a transformation matrix equal to

$$\boldsymbol{T} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.39}$$

If the directional spectrum $S(\omega, \theta)$ was known eq. (??) may be used to calculate $G_{mn}(\omega)$, e.g. by numerical integration.

Based on eq. (??) $\boldsymbol{C}$ and $\boldsymbol{Q}$ are identified as the real and the imaginary parts of $\boldsymbol{G}$. Finally $\boldsymbol{\kappa}_{\boldsymbol{A}\boldsymbol{A}^T}(\omega_l)$ can be determined from eq. (??). However, $S(\omega, \theta)$ is generally unknown. In the following a method will be presented which can be used to determine a directional spectrum expressed in standard form in terms of some unknown parameters. The method is known as the Maximum Likelihood (ML) method. As expressed earlier the elements in $\boldsymbol{A}$ have a joint normal distribution. The general expression for a density function of a vector $\boldsymbol{A}$ with $2M$ joint Gaussian elements having a mean value vector $\boldsymbol{E}(\boldsymbol{A}) = \boldsymbol{0}$ is

$$p_{\boldsymbol{A}}(\boldsymbol{a}) = \frac{1}{\left(\sqrt{2\pi}\right)^{2M} |\boldsymbol{\kappa}_{\boldsymbol{A}\boldsymbol{A}^T}(\tau)|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\, \boldsymbol{a}^T \boldsymbol{\kappa}^{-1}_{\boldsymbol{A}\boldsymbol{A}^T}(\tau) \boldsymbol{a}\right) \tag{3.40}$$

where $|\boldsymbol{\kappa}_{\boldsymbol{A}\boldsymbol{A}^T}|$ and $\boldsymbol{\kappa}^{-1}_{\boldsymbol{A}\boldsymbol{A}^T}$ are the determinant and the inverse matrix of $\boldsymbol{\kappa}_{\boldsymbol{A}\boldsymbol{A}^T}$ respectively.

If $S(\omega, \theta)$ was given eq. (??) could be used to calculate the probability of the observed realisation $\boldsymbol{a}$ of $\boldsymbol{A}$, where $\boldsymbol{a}$ represents the actual Fourier coefficients obtained from a given time-series. Since $S(\omega, \theta)$, or some parameters in $S(\omega, \theta)$, are unknown, a Likelihood function, $L(\cdot)$, will be formulated. Expressed in terms of the Likelihood function the unknown parameters in $S_\eta(\omega, \theta)$ are determined as the values corresponding to the maximum value of L.

In his article Isobe (1990) uses $\mathcal{L}$ time-series at each of the M locations $\boldsymbol{x}_p$, $p = 1, 2, \ldots, M$. Based on each of the $\mathcal{L}$ time-series an estimate $\boldsymbol{a}^{(l)}$, $l = 1, 2, \ldots, \mathcal{L}$, of $\boldsymbol{A}^{(l)}$ is obtained. The probability of $\boldsymbol{a}^{(l)}$ is given by eq. (??). Assuming the $\mathcal{L}$ observations to be independent the joint probability for obtaining exactly the $\mathcal{L}$ observed estimates $\boldsymbol{a}^{(l)}$ is given as $p_{\boldsymbol{A}}\left(\boldsymbol{a}^{(1)}\right) \cdot p_{\boldsymbol{A}}\left(\boldsymbol{a}^{(2)}\right) \cdot \ldots \cdot p_{\boldsymbol{A}}\left(\mathbf{a}^{(\mathcal{L})}\right)$. Therefore Isobe (1990) suggested a Likelihood function, $L(\cdot)$, as the $\mathcal{L}$'th root of this product, i.e.

$$
\begin{aligned}
L\left(\mathbf{a}^{(1)}, \ldots, \boldsymbol{a}^{(\mathcal{L})}, \boldsymbol{G}\right) &= \left\{p_{\boldsymbol{A}}\left(\boldsymbol{a}^{(1)}\right) \cdot \ldots \cdot p_{\boldsymbol{A}}\left(\boldsymbol{a}^{(\mathcal{L})}\right)\right\}^{1/\mathcal{L}} \\
&= \frac{1}{(\Delta\omega)^M \sqrt{\det(\boldsymbol{\Omega})}} \exp\left(-\frac{1}{2} \sum_{h=1}^{2M} \sum_{l=1}^{2M} \Omega_{hl}^{-1} \tilde{\Omega}_{lh}\right)
\end{aligned}
\tag{3.41}
$$

where

$$
\tilde{\Omega}_{lh} = \frac{2\pi}{\Delta\omega \cdot \mathcal{L}} \sum_{m=1}^{\mathcal{L}} a_{ml} a_{mh} \tag{3.42}
$$

where $\tilde{\boldsymbol{\Omega}}$ is the measured cross spectral density matrix.

Eq. (??) represents the probability of obtaining exactly the estimates $\boldsymbol{a}^{(l)}$, $l = 1, 2, \ldots, \mathcal{L}$. The unknown quantity in eq. (??) is $\boldsymbol{G}$ (or $S(\omega, \theta)$).

The optimal choice of $S(\omega, \theta)$ or (in practice) the unknown parameters in $S(\omega, \theta)$ are determined in order to maximise $L(\cdot)$, i.e. the optimal parameters maximise the probability of obtaining exactly the observed Fourier-coefficients.

3.4 The Bayesian Directional spectrum estimation Method

The Bayesian Directional spectrum estimation Method, BDM, is in principle similar to MLM. However, BDM makes use of a Bayesian approach in order to estimate the most likely estimate of the directional spectrum.

BDM has been presented by Hashimoto et al. (1987). It is assumed, that the directional spreading function $H(\omega, \theta)$ can be expressed as a piecewise-constant function, which takes only positive values. The directional spreading function is discretized into K intervals. Defining $x_l(\omega)$ as

$$x_l(\omega) = \ln H(\omega, \theta_l) \qquad l = 1, 2, ..., K \qquad K\Delta\theta = 2\pi \tag{3.43}$$

$H(\omega, \theta)$ can be approximated to

$$H(\omega, \theta) \simeq \sum_{l=1}^{K} \exp(x_l(\omega)) I_l(\theta) \qquad I_l = \begin{cases} 1 & (l-1)\Delta\theta \leq \theta < k\Delta\theta \\ 0 & \text{otherwise} \end{cases} \tag{3.44}$$

The relationship between the cross-spectrum and the directional spectrum has been deducted to

$$S_{mn}(\omega) = \int_0^{2\pi} S(\omega, \theta) \exp(-ikr_{mn} \cos(\theta - \beta_{mn})) d\theta \tag{3.45}$$

Inserting the approximation eq. (??) and dividing by $S(\omega)$ leads to

$$\begin{aligned} \frac{S_{mn}(\omega)}{S(\omega)} &\simeq \int_0^{2\pi} \sum_{l=1}^{K} \exp(x_l(\omega)) I_l(\theta) \exp(-ikr_{mn} \cos(\theta - \beta_{mn})) d\theta \\ &\simeq \sum_{l=1}^{K} \exp(x_l(\omega)) \exp(-ikr_{mn} \cos(\theta_l - \beta_{mn})) \Delta\theta \end{aligned} \tag{3.46}$$

If M wave gauges are available, eq. (??) can be applied to $N = M(M+1)/2$ spectra of which M will be autospectra. However, considering the N complex equations as two separate equations, i.e. the real term and the imaginary term, $2N$ real equations are obtained. Expressing these in an arbitrary order eq. (??) can be rearranged to

$$S_j(\omega) = \sum_{l=1}^{K} \exp(x_l(\omega)) \alpha_{jl}(\omega) + \varepsilon_j(\omega) \quad j = 1, 2, \ldots, M \tag{3.47}$$

where an error $\varepsilon_{mn}(\omega)$ of the cross-spectrum $S_{mn}(\omega)$ has been added, and

$$\alpha_{jl}(\omega) = \frac{\exp(-ikr_{mn} \cos(\theta_l - \beta_{mn})) \Delta\theta}{\sqrt{S_{mm}(\omega) S_{nn}(\omega)}} \tag{3.48}$$

$$S_j(\omega) = \frac{S_{mn}(\omega)}{S(\omega) \sqrt{S_{mm}(\omega) S_{nn}(\omega)}} \tag{3.49}$$

$$\varepsilon_j(\omega) = \frac{\varepsilon_{mn}(\omega)}{\sqrt{S_{mm}(\omega) S_{nn}(\omega)}} \tag{3.50}$$

where it is suggested that $1 \leq j \leq N$ denote real parts and $N < j \leq 2N$ denote imaginary parts. However, the number of equations involved can be altered at will.

It is assumed that $\varepsilon_j : N(0;\sigma^2)$, hence when omitting the argument ω

$$\varepsilon_j = S_j - \sum_{l=1}^{K} \exp(x_l)\alpha_{jl} \qquad\qquad : N(0;\sigma^2)$$

S_j's and α_{jl}'s are given. σ^2 and x_l's are to be estimated.

The probability density function for ε_j is

$$p(\varepsilon_j) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\varepsilon_j^2}{2\sigma^2}\right)$$

The likelihood function is then

$$\begin{aligned} L(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{2N}; \sigma) &= \prod_{j=1}^{2N} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\varepsilon_j^2}{2\sigma^2}\right) \\ &= \frac{1}{(2\pi\sigma^2)^N} \exp\left(-\sum_{j=1}^{2N} \frac{\varepsilon_j^2}{2\sigma^2}\right) \\ &= \frac{1}{(2\pi\sigma^2)^N} \exp\left(\frac{-1}{2\sigma^2} \sum_{j=1}^{2N} \left(S_j - \sum_{l=1}^{K} \exp(x_l)\alpha_{jl}\right)^2\right) \quad (3.51) \\ L(x_1, x_2, ..., x_K; \sigma) &= L(\varepsilon_1, \varepsilon_2, ..., \varepsilon_{2N}; \sigma) \quad (3.52) \end{aligned}$$

for the given $S_j,\ j = 1, 2, ..., 2N$.

As the estimate of $H(\omega, \theta)$ becomes smoother, it is assumed, that the differences of second order of $x_l - 2x_{l-1} + x_{l-2}$ decrease, i.e. the value

$$\sum_{l=1}^{K} (x_l - 2x_{l-1} + x_{l-2})^2 \quad (3.53)$$

where $x_0 = x_K$ and $x_{-1} = x_{K-1}$, decreases as well.

Maximising the likelihood eq. (??) and minimising eq. (??) lead to the estimate which also maximises

$$\ln L(x_1, x_2, ..., x_K; \sigma) - \frac{u^2}{2\sigma^2} \sum_{l=1}^{K} (x_l - 2x_{l-1} + x_{l-2})^2$$

where u is introduced as a hyperparameter. Further applying the exponential function leads to the alternative expression

$$L(x_1, x_2, ..., x_K; \sigma) \exp\left(-\frac{u^2}{2\sigma^2} \sum_{l=1}^{K} (x_l - 2x_{l-1} + 2x_{l-2})^2\right) \quad (3.54)$$

At this stage the Bayesian approach is introduced utilising $p(y|\boldsymbol{x}) \propto L(y|\boldsymbol{x})p(y)$. When normalised the second term in eq. (??) can be regarded as the joint distribution of $\boldsymbol{x} = (x_1, x_2, ..., x_K)$

$$p(\boldsymbol{x}|u^2, \sigma^2) = \left(\frac{u}{\sqrt{2\pi}\sigma}\right)^K \exp\left(-\frac{u^2}{2\sigma^2}\sum_{l=1}^{K}(x_l - 2x_{l-1} + x_{l-2})^2\right) \tag{3.55}$$

This correspond to the prior distribution and is known, when an estimate of $\boldsymbol{x}$ is given and u and σ have been estimated. Maximising eq. (??) leads to the posterior distribution, when inserting into $p(y|\boldsymbol{x}) \propto L(y|\boldsymbol{x})p(y)$, i.e.

$$p_{post}(\boldsymbol{x}|u, \sigma^2) \propto L(\boldsymbol{x}, \sigma^2)p_{prior}(\boldsymbol{x}|u, \sigma^2) \tag{3.56}$$

Hence minimising the following quantity determine the value of $\boldsymbol{x}$, which maximises eq. (??)

$$\frac{1}{2\sigma^2}\sum_{j=1}^{2N}\left(S_j - \sum_{l=1}^{K}\exp(x_l)\alpha_{jl}\right)^2 + \frac{u^2}{2\sigma^2}\left(\sum_{l=1}^{K}(x_l - 2x_{l-1} + x_{l-2})^2\right)$$

I.e. σ can be omitted reducing the above quantity to

$$\sum_{j=1}^{2N}\left(S_j - \sum_{l=1}^{K}\exp(x_l)\alpha_{jl}\right)^2 .+ u^2\left(\sum_{l=1}^{K}(x_l - 2x_{l-1} + x_{l-2})^2\right) \tag{3.57}$$

During the derivation of BDM it has been presumed that

- All values in the spreading functions are larger than zero
- The spreading functions are smooth (the smoothness being dependent on one parameter)
- The errors on the spectral estimates are outcomes of a Gaussian distribution function

The BDM turns out to be a very useful and reliable method for estimation of directional wave spectra. Some inaccuracy arises when a reflected wave system is involved, but it is still sufficiently reliable. Further it is rather easy to implement numerically.

3.5 References

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4 Tests of the presented methods

In principle all the methods for estimating reflected waves should be tested upon:

- Ability to process data correctly
- Robustness to possible errors
- Reliability of results from real wave data

A fully systematic test series of the methods for estimating reflected waves is under preparation at Aalborg University.

The following chapter will present the tests performed so far at Aalborg University within the frames of this MAST2-project.

4.1 Ability to process data

In this section the performance of the methods are evaluated with numerical data in situations where the reflection is known, and no possible noise (errors) on the data are included.

4.1.1 MLM utilising standard spectra

In the numerical tests performed, the generated data were simultaneous realizations of surface elevation time series recorded in a CERC5 wave gauge array with a radius of 1.0 m positioned at a waterdepth of 4.0 m, see fig. ??.

Two tests were performed: One in which no reflection occurred and one in which a wall with a reflection coefficient of r=0.5 was positioned at x=0.0 m, see fig. ??.

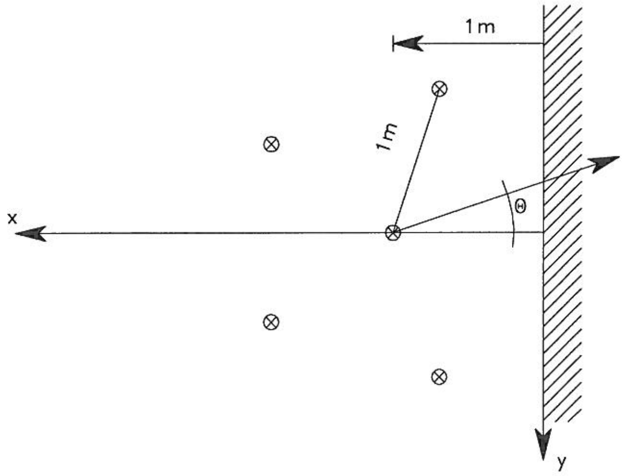

Figure 4.1: Wave gauge arrangement for numerical tests.

In both tests, the incident wave fields were irregular waves corresponding to a Pierson-Moskowitz spectrum ($f_p = 0.5Hz, H_s = 0.5m$) with a Mitsuyasu-type spreading function ($\theta_0 = \pi/3, s = 6$). A sample frequency of $f_s = 4Hz$ was applied.

Measured cross covariance matrices were determined from a total of 45 surface elevation subseries each of length T=128 s.

The estimated value of the autospectral density $S(\omega)$ and the maximum likelihood estimates of the parameters θ_0, s and r are given in fig. ?? and fig. ??. For comparison, the target values are plotted.

In both tests, the estimated values of the parameters are in good agreement with the target values. However, in the test involving reflection the method has some problems separating incident and reflected waves at some frequencies resulting in larger estimated values of the autospectral density, see fig. ?? compared to the results obtained without reflection, see fig. ??.

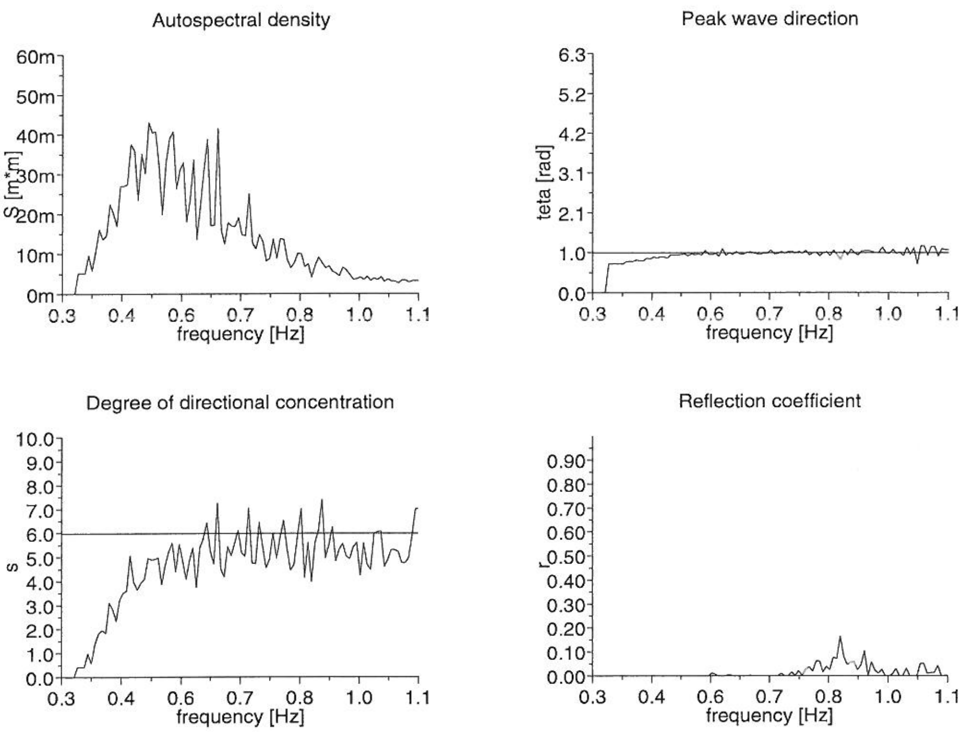

Figure 4.2: Numerical test results (no reflection).

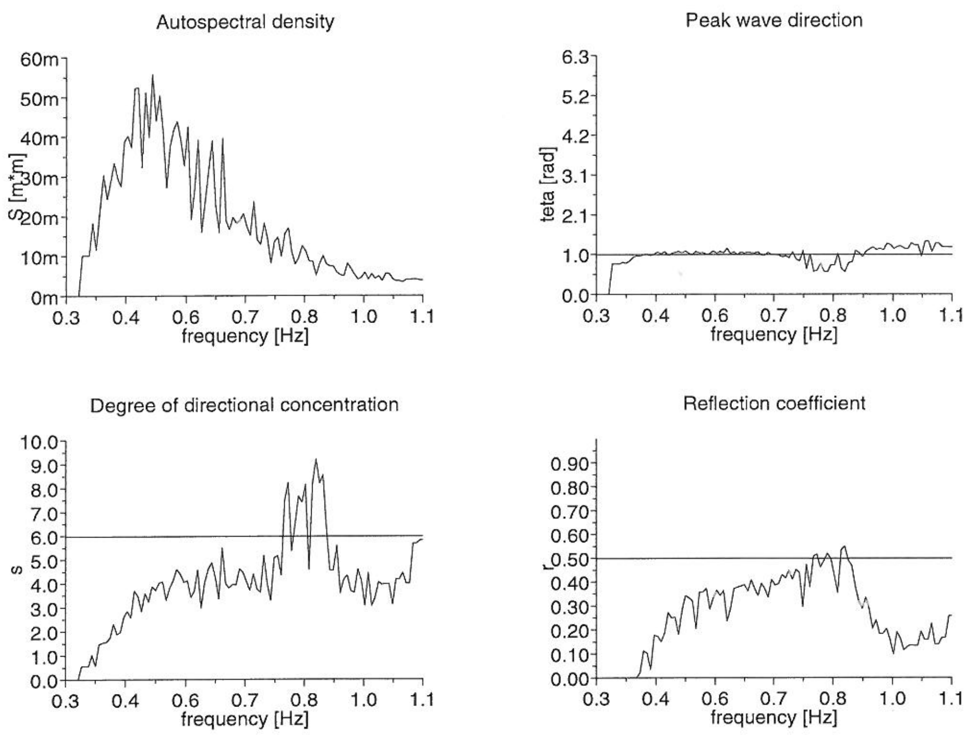

Figure 4.3: Numerical test results (with reflection).

4.1.2 BDM

The BDM method has been tested utilising numerical simulations of an uni-modal wave field and a bi-modal wave field.

Only the bi-directional wave field test is shown, see fig. ??. In this case it is seen that both peaks are estimated well. Though the narrow peak is estimated most accurate.

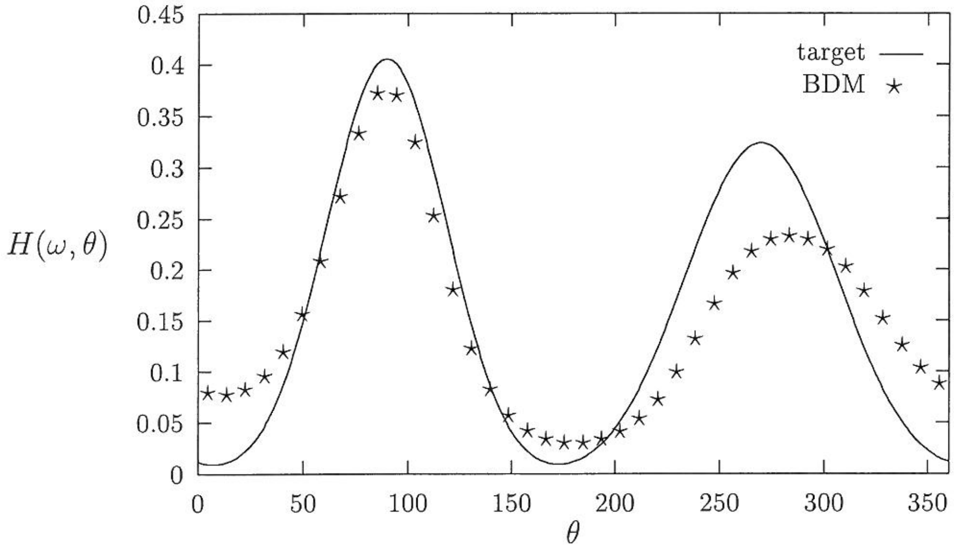

Figure 4.4: Test with simulated bi-directional wave field.

Data. Test 1.	
Date of sample	070893
Depth of water	$200m$
Radius of array	$26m$
Sample frequency	$4Hz$
Duration	$720sec.$
Quantity	Elevation
Autospectrum	P–M, $T_p = 10s,\ H_s = 4.0m$
Directional spreading function	Mits., $s_1 = 5, s_2 = 8, \theta_1 = -90°, \theta_2 = 90°$

4.2 Robustness to possible errors

In order to evaluate the performance of the proposed methods for estimating reflection, the methods are tested upon numerical data, where possible errors are introduced.

The possible errors are:

- Random noise on signals
- Non-linear effects in the waves
- Phase locked waves
- Three-dimensional waves in cases where a two-dimensional method is used
- Divergens in the estimation due to numerical problems in solving the system of equations

4.2.1 Random noise on signals

The following section describes a numerical test, where the elevations are calculated for monochromatic waves according to the shown equation:

$$\eta(x,t) = a_I \cos(kx - \omega t) + \alpha a_I \cos(kx + \omega t) + random \cdot \beta \cdot a_I$$

where α is reflection coefficient
β is noise coefficient
$random$ is a random number in $[-1;1]$

Goda & Suzuki								
β	ω	Δt	a_I	α	$\tilde{\alpha}_{12}$	$\tilde{\alpha}_{13}$	$\tilde{\alpha}_{23}$	$\tilde{\alpha}_{average}$
%	rad/sec	sec	metre	%	%	%	%	%
10	2π	0.0625	0.10	50.00	50.88	50.92	50.24	50.68
10	2π	0.0625	0.10	10.00	11.03	10.98	10.14	10.72

Goda & Suzuki								
β	ω	Δt	a_I	α	$\tilde{\alpha}_{12}$	$\tilde{\alpha}_{13}$	$\tilde{\alpha}_{23}$	$\tilde{\alpha}_{average}$
%	rad/sec	sec	metre	%	%	%	%	%
30	2π	0.0625	0.10	50.00	52.26	52.34	51.13	51.91
30	2π	0.0625	0.10	10.00	12.78	13.65	11.57	12.67

Mansard & Funke						
β	ω	Δt	a_I	α	$\tilde{\alpha}_{123}$	$\tilde{\alpha}_{average}$
%	rad/sec	sec	metre	%	%	%
10	2π	0.0625	0.10	50.00	50.50	50.50
10	2π	0.0625	0.10	10.00	10.42	10.42

Mansard & Funke						
β	ω	Δt	a_I	α	$\tilde{\alpha}_{123}$	$\tilde{\alpha}_{average}$
%	rad/sec	sec	metre	%	%	%
30	2π	0.0625	0.10	50.00	51.49	51.49
30	2π	0.0625	0.10	10.00	11.32	11.32

where Δt is sampling interval
α is reflection coefficient
$\tilde{}$ indicates estimated

All tests were performed with water depth $d = 0.50$ metre with a 160 sec. long time series. Distances between wave gauges were $x_{1,2} = 0.25$ metre and $x_{1,3} = 0.60$ metre .

The tests show good resistance to random noise. Though, the method proposed by Mansard & Funke gives slightly better results than the method proposed by Goda & Suzuki.

The tests were repeated for 15 different wave periods and all tests showed the same tendency as described in the examples above.

4.2.2 Non-linear effects in the waves

All the described methods assume linear waves. In order to examine the effect from non-linearities a 2-order wave were generated. First with a free 2-harmonic eq. (??), and then with a bounded 2-harmonics eq. (??):

$$\begin{aligned}\eta(x,t) &= a_I\cos(kx-\omega t)+0.2\cdot a_I\cos(2k^*x-2\omega t)+\\ &\quad \alpha a_I\cos(kx+\omega t)+\alpha\cdot 0.2\cdot a_I\cos(2k^*x-2\omega t) \qquad (4.1)\\ \eta(x,t) &= a_I\cos(kx-\omega t)+0.2\cdot a_I\cos(2kx-2\omega t)+\\ &\quad \alpha a_I\cos(kx+\omega t)+\alpha\cdot 0.2\cdot a_I\cos(2kx-2\omega t) \qquad (4.2)\end{aligned}$$

where k is wave number corresponding to ω
k^* is wave number corresponding to $2\cdot\omega$

Goda & Suzuki								
					Free 2-harmonics		Bounded 2-harm	
ω	Δt	a_I	a_I^{2-harm}	α	$\tilde{\alpha}_{average}^{1-harm}$	$\tilde{\alpha}_{average}^{2-harm}$	$\tilde{\alpha}_{average}^{1-harm}$	$\tilde{\alpha}_{average}^{2-harm}$
rad/sec	sec	metre	metre	%	%	%	%	%
π	0.0625	0.10	0.02	50.00	50.04	49.78	50.12	61.10
π	0.0625	0.10	0.02	10.00	10.22	10.18	10.18	26.33

Mansard & Funke								
					Free 2-harmonics		Bounded 2-harm	
ω	Δt	a_I	a_I^{2-harm}	α	$\tilde{\alpha}_{average}^{1-harm}$	$\tilde{\alpha}_{average}^{2-harm}$	$\tilde{\alpha}_{average}^{1-harm}$	$\tilde{\alpha}_{average}^{2-harm}$
rad/sec	sec	metre	metre	%	%	%	%	%
π	0.0625	0.10	0.02	50.00			50.00	53.14
π	0.0625	0.10	0.02	10.00			10.00	19.42

4.2.3 Numerical problems

Numerical problems due to the discretisation of the signal and the succeeding FFT-analysis.

Timeseries were calculated from:

$$\eta(x,t) = a_I \cos(kx - \omega t) + \alpha a_I \cos(kx + \omega t) \tag{4.3}$$

where α is reflection coefficient

Goda & Suzuki							
ω	Δt	a_I	α	$\tilde{\alpha}_{12}$	$\tilde{\alpha}_{13}$	$\tilde{\alpha}_{23}$	$\tilde{\alpha}_{average}$
rad/sec	sec	metre	%	%	%	%	%
2π	0.0625	0.10	50.00	50.21	50.32	49.84	50.12
2π	0.0625	0.10	10.00	10.28	10.45	9.80	10.18
$0.98 \cdot 2\pi$	0.0625	0.10	50.00	48.92	49.78	53.00	49.20
$0.95 \cdot 2\pi$	0.0625	0.10	50.00	53.88	34.64	53.32	42.98

Mansard & Funke					
ω	Δt	a_I	α	$\tilde{\alpha}_{123}$	$\tilde{\alpha}_{average}$
rad/sec	sec	metre	%	%	%
2π	0.0625	0.10	50.00	50.00	50.00
2π	0.0625	0.10	10.00	10.00	10.00
$0.98 \cdot 2\pi$	0.0625	0.10	50.00	49.66	49.66
$0.95 \cdot 2\pi$	0.0625	0.10	50.00	48.70	48.70

All tests were performed with water depth $d = 0.50$ metre with a 80 sec. long time serie. The time serie were divided into 10 sub-series, which were cosine tapered. Distances between wave gauges were $x_{12} = 0.25$ metre and $x_{13} = 0.60$ metre.

In order to examine errors from comming from a wrong calibration constant on one or two of the wave gauges.

Timeseries were calculated from:

$$\begin{aligned}
\eta(x,t) &= a_I \cos(kx - \omega t) + \alpha a_I \cos(kx + \omega t) \\
\eta(x_1,t) &= 1.00 \cdot a_I \cos(kx_1 - \omega t) + 1.00 \cdot \alpha a_I \cos(kx_1 + \omega t) \\
\eta(x_2,t) &= 1.05 \cdot a_I \cos(kx_2 - \omega t) + 1.05 \cdot \alpha a_I \cos(kx_2 + \omega t) \\
\eta(x_3,t) &= 1.10 \cdot a_I \cos(kx_3 - \omega t) + 1.10 \cdot \alpha a_I \cos(kx_3 + \omega t)
\end{aligned} \tag{4.4}$$

Goda & Suzuki							
ω	Δt	a_I	α	$\tilde{\alpha}_{12}$	$\tilde{\alpha}_{13}$	$\tilde{\alpha}_{23}$	$\tilde{\alpha}_{average}$
rad/sec	sec	metre	%	%	%	%	%
2π	0.0625	0.10	50.00	49.61	55.53	51.30	52.15
2π	0.0625	0.10	10.00	9.80	17.57	11.80	13.06

Mansard & Funke					
ω	Δt	a_I	α	$\tilde{\alpha}_{123}$	$\tilde{\alpha}_{average}$
rad/sec	sec	metre	%	%	%
2π	0.0625	0.10	50.00	51.40	51.40
2π	0.0625	0.10	10.00	18.90	18.90

4.3 Reliability of results from real wave data

In this section it is wanted to test the performance of the methods for estimating reflection in cases with real wave data. This can be wave data from wave basins or desirable wave data from proto type measurements.

4.3.1 BDM

In fig. ?? a test from Aalborg University's 3D wave basin is shown.

The main direction of the incident waves is 290 degrees and the mean direction of the reflected waves should be 70 degrees.

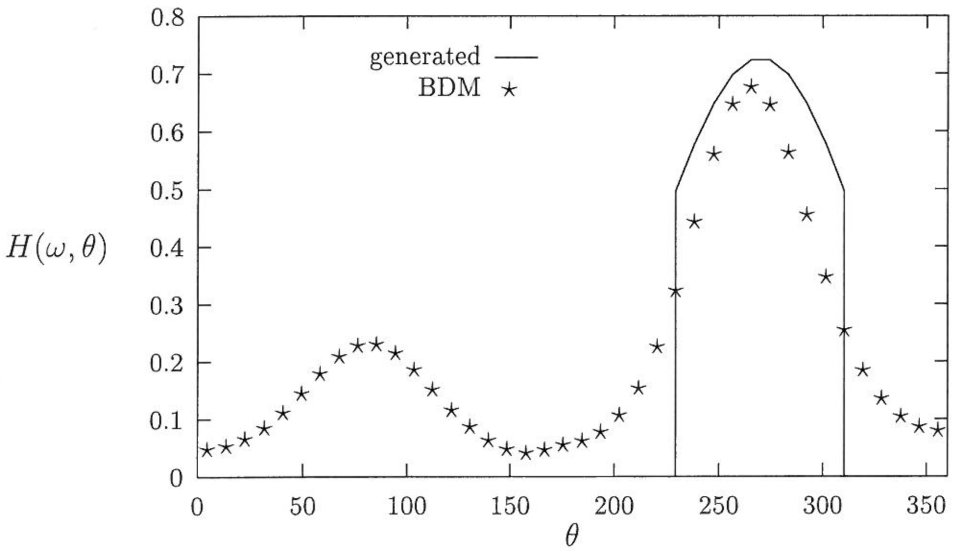

Figure 4.5: Test with reflected waves in 3D wave basin. Test 2.

Data. Test 2.	
Date of sample	090193
Depth of water	$0.6m$
Radius of array	$0.25m$
Sample frequency	$10Hz$
Duration	$720sec.$
Quantity	Elevation
Autospectrum	JONSWAP, $T_p = 0.8s,\ H_s = 0.1m$
Directional spreading function Mits., truncated $s_1 = 3, \theta_1 = -90°$	

The BDM method estimates the incident wave energy to be spread over a wider range. This is reasonable considering the conditions in the basin, i.e. limitations due to boundary conditions. The mean direction of the reflected waves though deviate about 20 degrees.

4.4 References

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Conclusion

Several different methods for estimating reflection have been presented and tested in the previous chapters.

The methods for estimating reflection are divided into groups:

- *Frequency domain* methods for 2-dimensional waves.
- *Time domain* methods for 2-dimensional waves.
- *Frequency domain* methods for 3-dimensional waves.

The *frequency domain* methods give the incident wave spectrum and the reflected wave spectrum. The *time domain* methods give the incident waves as function of time.

Some of the methods take into account the 3-dimensionality of the waves. If waves on location are 3-dimensional it is necessary to use a method for 3-dimensional waves, though because the methods for estimating reflection in 3-dimensional waves introduce the directional spreading function with many degrees of freedom, they become less statistically reliable.

In the MAST 2 project: *Full scale dynamic load monitoring of rubble mound breakwaters*, it is wanted to correlate measured pore pressure in the breakwater of Zeebrügge and forces on the armour layer blocks with the incident waves. Therefore a *time domain* method for estimating incident waves is preferable.

All the methods for 3-dimensional waves need an array of wave gauges e.g. 5 wave gauges, which is beyond the scope of the present MAST 2 project. However, this will not be a problem in Zeebrügge as the waves in front of the harbour have a 2-dimensional nature due to the relative mild sloping bottom in front of the harbour.

In Zeebrügge, it is recommended to use the *time domain* method presented in section (2.4) for estimating incident waves.

List of symbols

Variables and functions

a	:	amplitude
$\boldsymbol{a}$	:	realisation of $\boldsymbol{A}$
A, B	:	Fourier coefficients in cartesian form
$\boldsymbol{A}$	:	matrix of Fourier coefficients in polar form
$\boldsymbol{C}$	:	co-spectrum
d	:	waterdepth
f	:	frequency
f_p	:	peak frequency
G_{mn}	:	one-sided cross-spectral density matrix between η_m and η_n
H_s	:	significant waveheight
$H(\omega)$	:	frequency response filter
$H(\omega, \theta)$	:	directional spreading function
i	:	imaginary unit
k	:	wavenumber
k^*	:	wavenumber corresponding to 2ω
$\boldsymbol{k}$	:	wavenumber vector
$\mathcal{L}$	:	number of time-series recorded at a wave gauge
L	:	wavelength
$L(\cdot)$	:	likelihood function
M	:	number of wave gauges
$\mathcal{N}$	:	number of wave spectra in BDM
N	:	number of wave components
$N(\cdot)$	:	Gaussian distribution function
$p(\cdot)$	:	probability density function
$\boldsymbol{Q}$	:	quad-spectrum
r	:	distance between wave gauges
$S(\omega)$	:	autospectrum of elevation process
$S(\omega, \theta)$	:	directional spectrum
$\tilde{S}(\omega, \theta)$	:	estimated directional spectrum

$S_{mn}(\omega)$	:	cross-spectrum between η_m and η_n
t	:	time
Δt	:	sampling interval
T	:	wave period, length of time series
$\boldsymbol{T}$	:	transformation matrix
u	:	hyperparameter
$w(\cdot)$	:	window function
W	:	weight coefficient
x	:	coordinate in 1D system, discretisation of spreading function in BDM
$\boldsymbol{x}$	:	coordinates in 2D system
Δx	:	distance between two wave probes
Z	:	Fourier coefficients, polar form
α	:	reflection coefficient
β	:	mutual angel between wave gauges, noise coefficient
ε	:	error function
η	:	wave elevation
κ	:	proportionality factor
$\boldsymbol{\kappa}$	:	cross-covariance function matrix
λ	:	eigenvalue
Φ	:	phase
σ^2	:	variance
θ	:	direction of travel
θ_s	:	phaseshift at reflecting structure
ω	:	angular frequency
$\Delta\omega$	:	frequency step length
$\Omega(t)$	:	noise function
$\boldsymbol{\Omega}$	:	estimated cross-spectral density matrix
$\tilde{\boldsymbol{\Omega}}$	:	measured cross-spectral density matrix

Subscript

I	:	incident
N	:	noise
p	:	wave gauge number, peak
R	:	reflected
s	:	structure, sample

Superscript

T	:	transposed
$*$	:	complex conjugate, modified elevation signal
$'$	:	integration variable

Notation

Boldface letters denotes vectors and matrices. In general lowercase letters are vectors and uppercase letters are matrices. Indexed vector and matrix symbols denotes the specified element.